XV. *Researches on the Theory of Vortex Rings.*—Part II.

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The present communication forms a continuation of some researches the first part of which was published in Part I. of the Transactions for 1884.* In that paper was considered the case of a circular hollow with cyclic motion round it. In the following pages the more general case is investigated where the core is of different density from that of the surrounding fluid, has a hollow inside it, and circulations additional to that due to the rotational filaments actually present. The investigation is not merely one of mathematical interest, for the vortex atom theory of matter has—so far as it has yet been developed—shown such claims on our consideration that anything throwing light on it will be of value. The supposition of a dense core may possibly be necessary to account for the different masses of the various elements.

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* References to this are in square brackets, thus [I. 5]. [T. F.] refers to a paper on "Toroidal Functions" in Phil. Trans., Vol. 172 (1881), containing the theory of the functions used.

As soon as the existence of a core is postulated the ring at once becomes more complex, depending on the density (or even the arrangement of density) of its core, on its vorticity, and on the presence or absence of additional circulations. In what follows the vorticity has been taken uniform; this not only greatly simplifies the mathematical methods, but is also the case we should naturally choose first to investigate. In the general investigation the density is taken to be different from that of the surrounding fluid. The ring is supposed hollow, with an additional circulation round it, and another additional circulation round the outer boundary of the core. It is evident that the presence of the former circulation necessitates the perpetual existence of the hollow. It is shown that the presence of the latter circulation is necessary to render the ring stable when its density is greater than that of the rest of the fluid.

As in the former paper, the investigation is divided into three sections. The first is preliminary and deals with the necessary functions and their approximate values. The second is devoted to the consideration of the state of steady motion. Here the approximations are carried in the beginning so far as to include the second order of infinitesimals, but this necessitates in certain parts approximation to the fourth order. It has been carried to this order for future work; the reader however may, so far as the results of the present communication are concerned, without any loss of intelligibility, pass over the parts giving the calculation of the highest orders. The third section discusses the question of fluted vibrations, of pulsations, and of stability.

When the motion is steady the sectional centre* of the hollow lies outside that of the core. In general (§ 9) if C_1 is its position (with given outside boundary) when the inner additional circulation is very large, and C_2 when the same quantity is zero, C_1 is outside of C_2 and the position of C when the additional circulation is general, is the centre of gravity of masses proportional to the added circulation at C_1 and the circulation due to the core itself at C_2. When the hollow is just zero, the distance of C_2 from the sectional centre of the core bears to the sectional radius the ratio $5r/8a$ where r is the sectional radius, and a the radius of the ring. This, therefore, is the point where the hollow begins to form when the energy is sufficiently increased. If with the same outer boundary the mass of the core be lessened (or size of hollow increased) C_2 moves in and ultimately coincides with the centre of the outer section. The position of C_1 alters in the same manner, only in this case the hollow can never vanish.

If m be the volume of the core, Π the pressure at an infinite distance, μ the circulation when there are no additional ones, and d, d' the densities, then (§ 10) a hollow will begin to form when the radius of the ring is $4m\Pi/\mu^2(d+d')$. So long as the core is simply continuous the volume is constant, and therefore the sectional radius varies inversely as the square root of the radius of the ring. When there is no core it was shown in the former paper that the sectional radius of the hollow remained constant. In the general case, after a hollow is formed the sectional radius of the core changes more slowly, and the additional circulations add to this tendency. The outer section always decreases as the aperture increases, but when the hollow becomes large this decrease is very small, and the sectional radius of the core remains almost constant. The sectional radius of the hollow also increases with the aperture. In cases where a hollow begins to form the sectional radius at that time is equal to $\sqrt{1+\sigma}$ times its ultimate value; σ being the density of the core with respect to the surrounding fluid.

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* By sectional centre is meant the centre of the cross-section; by apertural centre is meant the centre of the aperture.

The expansibility of the ring due to the presence of a hollow has a marked effect on the variation of the velocity of translation with increasing aperture, the tendency being to make the variation smaller.

With an internal additional circulation the ring will possess internal energy comparable with that of the external fluid. It will, however, decrease as the whole energy is increased. This is of importance for the general theory of gases.

The fluted vibrations in general consist of two sets of two, travelling in opposite directions round the core, the modes being defined by the number of flutings. For a single continuous core there are two sets; for a hollow core without an internal additional circulation, there are three sets together with a standing wave (where the time of vibration is infinite); for two additional circulations we get four sets, the times being determined by a biquadratic, which I have not succeeded in solving in general terms. When there is no rotational core the motion is always stable. When there is a simple continuous core, whose density referred to the outer fluid is σ, and no additional inner circulation, the ratio of the outer circulation to that due to the core must be $> \sqrt{\left\{\frac{\sigma}{2}(2+\sigma)/(1+\sigma)\right\}}$. When there is no additional circulation, or no slip over the core, the ring cannot be always stable unless $\sigma < \sqrt{2}$. These conditions hold until a hollow has formed. When there is a hollow and no internal additional circulation, the simple ring usually considered is still stable. But if the core is denser than the surrounding fluid, it is always stable only when the outer additional circulation is larger than a certain critical value depending on the densities and the circulations. If it is less than this critical value the ring becomes unstable at some point as the aperture increases. When the density of the core is very large, of the order 10^p (p large) this critical value is $\sqrt{\frac{3}{2}}\,10^{p/3}$ times the circulation of the core itself.

The condition of stability when there is an inner additional circulation depends on the reality of the roots of a biquadratic equation, and the general conditions are not discussed, but the same property of the outer additional circulation preserving the stability clearly holds good. The motion is always stable for pulsations.

These conditions of stability only reach so far as fluted vibrations and pulsations are concerned. The question of the stability for twisted and beaded vibrations is not considered. J. J. THOMSON has proved that simply continuous rings of the same density as the rest of the fluid are stable for these kinds of vibrations. The general case yet remains to be investigated.

Some of the simpler results are here collected for the sake of reference.

$$\text{(distance of } C_1 \text{ from sectional centre)}/r = \frac{r}{4a}\,(1 - x + \log 1/x) \quad . \quad . \quad (38)$$

$$\text{(distance of } C_2 \text{ from sectional centre)}/r = \frac{r}{4a}\left\{\tfrac{5}{2} - \frac{3x}{2} - \frac{x}{1-x}\log 1/x\right\} \quad . \quad (38)$$

$$ar^2(1-x) = \frac{m}{2\pi^2}$$

$$\frac{r^2}{r_0^{\,2}}\,\mu_2^{\,2}\rho = \mu_2^{\,2}\rho - (\mu_1 + \mu')^2 + \frac{\mu_1^{\,2}}{x} + 2\mu'^2\,\frac{1 - x - x\log 1/x}{(1-x)^2} + 2\mu_1\mu'\,\frac{\log 1/x}{1-x} \quad . \quad . \quad (43)$$

$$\frac{a_1}{a}\,(1+\rho) = (1-x)\rho + (1+x) - \frac{2x}{1-x}\log 1/x \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

The general formula for the velocity of translation is given in eq. (44), the following are cases :—

No core

$$V = \frac{\mu_2}{4\pi a}\left(\log\frac{8a}{r} - \tfrac{1}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Continuous core

$$V = \frac{\mu}{4\pi a}\left(\log\frac{8a}{r} + \tfrac{5}{8}\right) + \frac{11}{32\pi a}\,\frac{\mu'^2}{\mu_2\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Ordinary ring

$$V = \frac{\mu}{4\pi a}\left(\log\frac{8a}{r} - \tfrac{1}{4}\right)$$

For hollow with no added circulations

$$V = \frac{\mu}{8\pi a}\left\{\log\frac{64a^2}{rr'} - \frac{1+x^2}{(1-x)^2}\log 1/x\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

where

μ' = circulation due to vortex filaments
μ_1 = added inner circulation
μ_2 = whole outer circulation
ρ = ratio of density of outside fluid to density of core
m = volume of core
r = outer sectional radius

r' = inner sectional radius
x = $(r'/r)^2$
r_0 = value of r when the aperture is infinite
a = radius of ring
a_1 = radius of ring when the hollow begins to form
r_1 = sectional radius " " " "
V = velocity of translation

Section I.—*Preliminary.*

1. The motion we are about to consider is one of steady motion in a fluid, part of which is rotational. If ψ denote the stream function, ω the angular rotation, and ρ, z, the cylindrical co-ordinates of a point, then

$$\frac{\delta^2\psi}{\delta z^2}+\frac{\delta^2\psi}{\delta\rho^2}-\frac{1}{\rho}\frac{\delta\psi}{\delta\rho}=2\omega\rho$$

Now when the motion is steady the rotations are so arranged that the vorticity is a function of the stream line or $2\omega/\rho=f(\psi)$. For steady motion then we have

$$\frac{\delta^2\psi}{\delta z^2}+\frac{\delta^2\psi}{\delta\rho^2}-\frac{1}{\rho}\frac{\delta\psi}{\delta\rho}=\rho^2 f(\psi)$$

Before, therefore, it is possible to discuss the properties of any vortex ring it is necessary to know its vorticity. The case considered in the following pages is that where the vorticity is constant. The methods developed will however apply to any cases where the motion is arranged in anchor-ring shells, the vorticity in any shell being constant. Here then $f(\psi)$ is a constant $=$A (say). A particular integral is then at once obtained, viz., $\psi=\frac{1}{8}\mathrm{A}\rho^4$, and the general solution becomes [I. § 1].

$$\psi=\tfrac{1}{8}\mathrm{A}\rho^4+\frac{1}{\sqrt{\mathrm{C}-c}}\Sigma_0^\infty(\mathrm{A}_n\mathrm{R}_n+\mathrm{B}_n\mathrm{T}_n)\cos(nv+\alpha_n)$$

Since the translatory motion is uniform the problem may be reduced to one of steady motion by impressing on every point a velocity equal and opposite to V, the velocity of translation. The stream function for the fluid outside the core will then be of the form

$$\psi_2=-\tfrac{1}{2}\mathrm{V}\rho^2+\frac{1}{\sqrt{\mathrm{C}-c}}\Sigma_0^\infty\mathrm{A}_n{}'\mathrm{R}_n\cos nv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

whilst for the portion of fluid constituting the core it is of the form

$$\psi_1=\tfrac{1}{8}\mathrm{A}\rho^4+\frac{1}{\sqrt{\mathrm{C}-c}}\Sigma_0^\infty(\mathrm{A}_n\mathrm{R}_n+\mathrm{B}_n\mathrm{T}_n)\cos nv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We shall assume—an assumption to be justified by the result—that

$$(A_nR_n+B_nT_n)/(A_{n-1}R_{n-1}+B_{n-1}T_{n-1})$$

is of order k, or of the first order of small quantities by which the approximation proceeds. In this case A_n/A_{n-1} is of order k^2, and B_n/B_{n-1} of zero order, in other words A_n/A_0 of order k^{2n}, and B_n of the same order as B_0, but there is no means at present of determining the order of A_0, or A'_0 with reference to B_0.

2. The most general case considered is where the core is hollow, and has therefore an inner free surface, and an outer surface common with the fluid moving irrotationally. The cross sections of these surfaces will approximate to circles. Take the critical circle (radius a) of our curvilinear coordinates to be that belonging to the mean circle of the inner cross section, and for this mean circle let k be k_1. Its actual form can then be represented by the equation

$$k=k_1(1+\alpha_2\cos 2v+\alpha_3\cos 3v+\dots) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where α_n is of order k^n.

The outer surface can be represented by the form

$$k=k_2(1+\beta_1\cos v+\beta_2\cos 2v+\beta_3\cos 3v+\dots) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where, as in the former case, β_n is of order k^n.

It might be thought that it would be possible to represent the outer surface by an equation of the first form, and the inner surface by one of the second form. But we have no right to do this, for the inner surface might not contain the critical circle, and it would then be impossible to represent it by the second form. To assume that form for it would, therefore, be equivalent to assuming that the inner surface contains the critical circle of the outer. Now, the outer must evidently contain that of the inner, hence the equations above given can actually represent them.

The mean circle approximating most closely to the outer surface will not be that represented by k_2, but one which does not belong to the system k at all. It will be necessary to know the distance of its centre from that of the inner mean circle. Now [T. F., Eq. 6] if R, r, denote the radius of the axial circle, and of the cross-section respectively of a tore (u),

$$R/r=C \qquad a/r=S$$

Hence for the inner surface

$$\left.\begin{array}{ll}\text{radius of cross-section} & =a/S\ =2ak_1 \\ \text{radius of ring} & =aC/S=a(1+2k_1^2)\end{array}\right\}\text{to second order.}$$

To determine the similar quantities for the outer surface let it meet the plane of the ring in the points (k', k''), then

$$k'=k_2(1-\beta_1) \qquad k''=k_2(1+\beta_1) \qquad \text{(to the second order).}$$

Hence

$$\begin{aligned}\text{radius of cross-section} &= \tfrac{1}{2}\{(\mathrm{R}''+r'')-(\mathrm{R}'-r')\}\\ \text{radius of ring} &= \tfrac{1}{2}\{(\mathrm{R}'-r')+(\mathrm{R}''+r'')\}\end{aligned}$$

From these it follows easily that

$$\left.\begin{aligned}\text{radius of section} &= 2ak_2\\ \text{radius of ring} &= a(1+2k_2^2+2k_2\beta_1)\\ \text{and distance between centres of mean cross sections}&\\ &= 2a(k_2^2-k_1^2+k_2\beta_1)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

the centre of the outer lying outside that of the inner.

The volume of the core will be

$$\iint 2\pi\rho\frac{dn}{du}\cdot\frac{dn'}{dv}dudv$$

taken over the section. Let us first find the volume of the surface bounded by

$$k=k_2(1+\beta_1\cos v\ldots)$$

Volume

$$\begin{aligned}&=2\pi a^3\int_0^{2\pi}\int_u^\infty\frac{\mathrm{S}}{(\mathrm{C}-c)^3}dudv\\ &=-\pi a^3\int_0^{2\pi}\left[\frac{1}{(\mathrm{C}-c)^2}\right]_u^\infty dv\\ &=\pi a^3\int_0^{2\pi}\frac{dv}{(\mathrm{C}-c)^2}\end{aligned}$$

Now

$$\begin{aligned}\frac{1}{(\mathrm{C}-c)^2}&=\frac{1}{\mathrm{C}^2}\left(1+\frac{2c}{\mathrm{C}}+\frac{3c^2}{\mathrm{C}^2}+\ldots\right)\\ &=4k^2(1+4k^2+4k\cos v+6k^2\cos 2v+\ldots)\end{aligned}$$

Along the boundary $k=k_1(1+\beta_1\cos v+\beta_2\cos 2v)$.

Therefore to the second order in the bracket

$$\frac{1}{(\mathrm{C}-c)^2}=4k_1^2\{1+4k_1^2+\tfrac{1}{2}\beta_1^2+6\beta_1k_1+p\cos v+q\cos 2v+\ldots\}$$

Hence volume

$$=8\pi^2a^3k_1^2(1+4k_1^2+\tfrac{1}{2}\beta_1^2+6\beta_1k_1)$$

or volume of core

$$=8\pi^2a^3\{k_2^2-k_1^2+4(k_2^4-k_1^4)+\tfrac{1}{2}\beta_1^2k_2^2+6\beta_1k_2^3\}=m \text{ (say)}. \quad . \quad . \quad . \quad (6)$$

i.e., to lowest order

$$m=8\pi^2a^3(k_2^2-k_1^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

3. The circulation of the ring, taking the velocity through the ring as positive, is

$$-2\iint\tfrac{1}{2}\mathrm{A}\rho\frac{dn}{du}\cdot\frac{dn'}{dv}dudv$$

taken over the cross section. Hence by what has gone before, the circulation due to the actual vortex filaments of the core is

$$\mu'=-4\pi\mathrm{A}a^3\{k_2^2-k_1^2+4(k_2^4-k_1^4)+\tfrac{1}{2}\beta_1^2k_2^2+6\beta_1k_2^3\} \quad . \quad . \quad . \quad . \quad (8)$$

The outside stream function is

$$\psi_2=-\tfrac{1}{2}\mathrm{V}\rho^2+\frac{1}{\sqrt{\mathrm{C}-c}}\Sigma_0^\infty\mathrm{A}_n'\mathrm{R}_n\cos nv$$

Hence the circulation of the fluid outside the ring is [I., Eq. 4]

$$\mu_2=-\frac{\pi\sqrt{2}}{a}(\mathrm{A}_0'+\mathrm{A}_1'+\mathrm{A}_2'+\ldots) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Again taking the circulation round the inner surface of the core, the circulation in the core additional to that due to its own rotation is

$$\mu_1=-\frac{\pi\sqrt{2}}{a}(\mathrm{A}_0+\mathrm{A}_1+\mathrm{A}_2+\ldots)-4\pi\mathrm{A}a^3(k_1^2+4k_1^4) \quad . \quad . \quad . \quad . \quad (10)$$

For the sake of greater generality we shall suppose circulations, additional to that due to the core, as existing in the outer irrotationally moving fluid, and in the core itself. In the case where there is none added to the outside fluid

$$\mu_2=\mu_1+\mu'$$

whence to the lowest order of small quantities

$$\mathrm{A}_0'=\mathrm{A}_0+\frac{4\mathrm{A}a^4}{\sqrt{2}}k_2^2$$

If there are no added rotations A_0, A'_0 will therefore be of order k^2 with reference to Aa^4. By always taking them to be of this order therefore the approximation will still be true when there are added circulations.

4. It will be convenient here to consider the equation giving the pressure at any part of the core, including the case where small vibrations are superposed on the steady motion. The equations of motion are, d being the density, and taking the motion in the plane of z, x,

$$-\frac{1}{d}\frac{\delta p}{\delta x}=\frac{\delta u}{\delta t}+u\frac{\delta u}{\delta x}+w\frac{\delta u}{\delta z}$$

$$-\frac{1}{d}\frac{\delta p}{\delta z}=\frac{\delta w}{\delta t}+u\frac{\delta w}{\delta x}+w\frac{\delta w}{\delta x}$$

$$\frac{\delta u}{\delta z}-\frac{\delta w}{\delta x}=2\omega=-Ax$$

Therefore, integrating along a line in the plane of x, z,

$$-\frac{1}{d}\left(\frac{\delta p}{\delta x}dx+\frac{\delta p}{\delta z}dz\right)=\frac{\delta}{\delta t}(udx+wdz)+u\left(\frac{\delta u}{\delta x}dx+\frac{\delta u}{\delta z}dz\right)$$
$$+w\left(\frac{\delta w}{\delta x}dx+\frac{\delta w}{\delta z}dz\right)+Ax(udz-wdx)$$

or if v denote the velocity at any point and v' the velocity along the line of integrations,

$$-\frac{1}{d}\frac{\delta p}{\delta s}=\frac{\delta v'}{\delta t}+\frac{1}{2}\frac{\delta(v^2)}{\delta s}-A\frac{\delta\psi}{\delta s}$$

Therefore

$$\frac{p}{d}=f(t)-\dot{\phi}-\tfrac{1}{2}v^2+A\psi \quad \ldots\ldots\ldots\ldots \quad (11)$$

where ϕ is the flow along any line in a plane through the axis of z up to the point in question. It is a function depending in general on the path of integration, but $\dot{\phi}$ is independent of this (is in fact due to the added irrotational motion) and is single valued when ψ is so. When ψ is many-valued (as, for instance, in the case of pulsations), $\dot{\phi}-A\psi$ is single-valued.

Section II.—*Steady motion.*

5. Suppose the approximations are carried so far as to include terms in cos nv in the stream functions and equations to the bounding surfaces. The constants to be determined will then be V, the $n-1$ quantities α, the n quantities β, and the coefficients A'_n, A_n, B_n, $3(n+1)$ in number—or in all $5n+3$ quantities. We may

regard A'_0, A_0 as determined by (eq. 9, 10) when the circulations are given, and B_0 will be determined by the fact that the pressure along the free surface is zero, whilst it is Π at an infinite distance. There remain therefore $5n$ constants to be determined. The conditions to be satisfied are the following:—

(1) ψ_2 must be constant along the outer boundary of the core. This gives n equations, viz., by equating to zero the coefficients of cos v .... cos nv along the surface.

(2) ψ_1 constant along both the outer and inner boundaries of the core. This gives $2n$ equations.

(3) The pressure must be the same on both sides of the outer boundary. This gives n equations.

(4) The pressure along the inner surface must be constant. This gives also n equations. Hence on the whole the surface conditions give $5n$ equations, sufficient therefore to determine the $5n$ constants. It is therefore possible to approximate to any order desired.

It has already been noticed that A'_0, A_0 are to be regarded as of the order k^2 with respect to Aa^4. It will be seen later that when this is the case the B are of zero order with respect to the same quantity. Hence, if in the approximations account is taken of A_2, the term in which B occurs must be carried as far as $B_4 \cos 4v$. It will be necessary therefore to carry the latter terms to the fourth order, except in the case where the added circulation is very large compared with that due to the actual rotational motion. This case is therefore a much easier one to discuss than the more general one. We proceed to apply the conditions given above to determine ψ_2, ψ_1. Our method of procedure will be first to express the functions in terms of k, and the cosines of multiples of v; then to substitute the value of k along the surface, and reduce the expressions to a series of cosines of multiples of v, whose coefficients are functions of k_1 or k_2 as the case may be. The conditions above are then applicable at once.

6. *The function* ψ_2. It has been seen (1) that

$$\psi_2 = -\tfrac{1}{2}V\rho^2 + \frac{1}{\sqrt{C-c}}\Sigma_0^\infty A'_n R_n \cos nv$$

Now,

$$\tfrac{1}{2}V\rho^2 = \tfrac{1}{2}a^2V\left(\frac{S}{C-c}\right)^2 = \tfrac{1}{2}a^2V\left\{\frac{1-k^2}{1+k^2-2kc}\right\}^2$$

$$= \tfrac{1}{2}a^2V(1+2k^2+4k\cos v+6k^2\cos 2v)$$

For later purposes the value of $(C-c)^{-\frac{1}{2}}$ must be carried to the fourth order, and then

$$\begin{aligned}\frac{1}{\sqrt{(\mathrm{C}-c)}}&=\frac{\sqrt{(2k)}}{\sqrt{(1+k^2-2kc)}}\\&=\sqrt{(2k)}\Big\{1+\tfrac{1}{4}k^2+\tfrac{9}{64}k^4+(k+\tfrac{3}{8}k^3)\cos v+\Big(\frac{3k^2}{4}+\tfrac{5}{16}k^4\Big)\cos 2v\\&\qquad+\tfrac{5}{8}k^3\cos 3v+\tfrac{35}{64}k^4\cos 4v\Big\}\end{aligned}\quad(12)$$

Substituting the values of R given in [I., Eq. 11],

$$\begin{aligned}&\mathrm{A}'_0\mathrm{R}_0+\mathrm{A}'_1\mathrm{R}_1\cos v+\mathrm{A}'_2\mathrm{R}_2\cos 2v\\&\quad=-k^{-\frac{1}{2}}\{\tfrac{1}{2}\mathrm{L}-1+\tfrac{1}{8}(\mathrm{L}+1)k^2\}\mathrm{A}'_0+\tfrac{1}{2}k^{-\frac{3}{2}}\{1-\tfrac{3}{2}(\mathrm{L}-\tfrac{1}{2})k^2\}\mathrm{A}'_1\cos v\\&\qquad+k^{-\frac{5}{2}}(1-\tfrac{5}{4}k^2)\mathrm{A}'_2\cos 2v\end{aligned}$$

Putting in these values and reducing, it will be found that

$$\left.\begin{aligned}\frac{1}{\sqrt{2}}\psi_2=&-\tfrac{1}{2}\mathrm{A}'_0\{\mathrm{L}-2+\tfrac{1}{4}(2\mathrm{L}-1)k^2\}+\tfrac{1}{4}\mathrm{A}'_1-\frac{a^2\mathrm{V}}{2\sqrt{2}}\Big(1+2k^2\Big)\\&+\Big\{-\tfrac{1}{2}\mathrm{A}'_0(\mathrm{L}-2)+\tfrac{1}{2}\frac{\mathrm{A}'_1}{k^2}-\frac{2a^2\mathrm{V}}{\sqrt{2}}\Big\}k\cos v\\&+\Big\{-\tfrac{3}{8}\mathrm{A}'_0(\mathrm{L}-2)+\tfrac{1}{4}\frac{\mathrm{A}'_1}{k^2}+\frac{\mathrm{A}'_2}{k^4}-\frac{3a^2\mathrm{V}}{\sqrt{2}}\Big\}k^2\cos 2v\end{aligned}\right\}\quad .\ .\ .\ (13)$$

Now, along the outer surface

$$k=k_2(1+\beta_1\cos v+\beta_2\cos 2v)$$

whence

$$\left.\begin{aligned}\mathrm{L}&=\mathrm{L}_2+\tfrac{1}{4}\beta_1^2-\beta_1\cos v+(\tfrac{1}{4}\beta_1^2-\beta_2)\cos 2v\\k^2&=k_2^2\{1+\tfrac{1}{2}\beta_1^2+2\beta_1\cos v+(2\beta_2+\tfrac{1}{2}\beta_1^2)\cos 2v\}\end{aligned}\right\}\quad .\ .\ .\ (14)$$

Suppose that when these values are substituted ψ_2 becomes $p_0+p_1\cos v+p_2\cos 2v$. This gives the value of ψ_2 along the outer surface, and since this is constant, p_1, p_2 both vanish.

Make these substitutions and reduce, then it will be found that

$$\left.\begin{aligned}\frac{1}{\sqrt{2}}p_0=&-\tfrac{1}{2}\mathrm{A}'_0\Big\{\mathrm{L}_2-2+\tfrac{1}{4}(2\mathrm{L}_2-1)k_2^2\Big\}+\tfrac{1}{4}\mathrm{A}_1-\tfrac{1}{2}\frac{a^2\mathrm{V}}{\sqrt{2}}(1+2k_2^2)\\&-\tfrac{1}{2}\beta_1\Big\{\tfrac{1}{2}\mathrm{A}'_0(\mathrm{L}_2-3)k_2+\tfrac{1}{2}\frac{\mathrm{A}'_1}{k_2}+\tfrac{1}{4}\mathrm{A}'_0\beta_1+\frac{2a^2\mathrm{V}}{\sqrt{2}}k_2\Big\}\\\frac{p_1}{\sqrt{2}}=&-\tfrac{1}{2}\mathrm{A}'_0(\mathrm{L}_2-2)k_2+\tfrac{1}{2}\frac{\mathrm{A}'_1}{k_2}-\frac{2a^2\mathrm{V}}{\sqrt{2}}k_2+\tfrac{1}{2}\beta_1\mathrm{A}'_0=0\\\frac{p_2}{\sqrt{2}}=&-\tfrac{3}{8}\mathrm{A}'_0(\mathrm{L}_2-2)k_2^2+\tfrac{1}{4}\mathrm{A}'_1+\frac{\mathrm{A}'_2}{k_2^2}-\frac{3a^2\mathrm{V}}{\sqrt{2}}k_2^2-\tfrac{1}{2}(\tfrac{1}{4}\beta_1^2-\beta_2)\mathrm{A}'_0\\&+\tfrac{1}{2}\beta_1\Big\{-\tfrac{1}{2}\mathrm{A}'_0(\mathrm{L}_2-3)k_2-\tfrac{1}{2}\frac{\mathrm{A}'_1}{k_2}-\frac{2a^2\mathrm{V}}{\sqrt{2}}k_2\Big\}=0\end{aligned}\right\}\ .\quad(15)$$

Hence

$$A'_1=\frac{4a^2V}{\sqrt{2}}k_2^2+\left(L_2-2-\frac{\beta_1}{k_2}\right)A'_0k_2^2 \quad \ldots\ldots\ldots\ldots\ldots \quad (16)$$

$$\frac{A'_2}{k_2^4}=-\tfrac{1}{2}A'_0\frac{\beta_2}{k_2^2}+\tfrac{1}{8}A'_0(L_2-2)+\frac{2a^2V}{\sqrt{2}}+\tfrac{1}{2}\frac{\beta_1}{k_2}\left\{A'_0(L_2-2)+\frac{4a^2V}{\sqrt{2}}-\tfrac{1}{4}A'_0\frac{\beta_1}{k_2}\right\} \quad . \quad (17)$$

Now by (9)

$$A'_0+A'_1+A'_2=-\frac{\mu_2 a}{\pi\sqrt{2}}$$

Hence to the second order

$$A'_0\left\{1+\left(L_2-2-\frac{\beta_1}{k_2}\right)k_2^2\right\}=-\frac{\mu_2 a}{\pi\sqrt{2}}-\frac{4a^2V}{\sqrt{2}}k_2^2$$

or

$$A'_0=-\frac{\mu_2 a}{\pi\sqrt{2}}+\left\{\frac{\mu_2 a}{\pi\sqrt{2}}\left(L_2-2-\frac{\beta_1}{k_2}\right)-\frac{4a^2V}{\sqrt{2}}\right\}k_2^2 \quad \ldots\ldots \quad (18)$$

Again, differentiating (13) with respect to k

$$\left.\begin{aligned}\frac{1}{\sqrt{2}}k\frac{\delta\psi_2}{\delta k}=&\tfrac{1}{2}A'_0\{1-(L-1)k^2\}-\frac{2a^2V}{\sqrt{2}}k^2\\&+\left\{-\tfrac{1}{2}A'_0(L-3)-\tfrac{1}{2}\frac{A'_1}{k^2}-\frac{2a^2V}{\sqrt{2}}\right\}k\cos v\\&+\left\{-\tfrac{3}{8}A'_0(2L-5)-\frac{2A'_2}{k^4}-\frac{6a^2V}{\sqrt{2}}\right\}k^2\cos 2v\end{aligned}\right\} \quad \ldots\ldots \quad (19)$$

Along the surface this becomes

$$\left.\begin{aligned}\frac{1}{\sqrt{2}}k\frac{\delta\psi_2}{\delta k}=&\tfrac{1}{2}A'_0\{1-(L_2-1)k_2^2\}\\&-\frac{2a^2V}{\sqrt{2}}k_2^2+\tfrac{1}{2}\beta_1\left\{-\tfrac{1}{2}A'_0(L_2-4)k_2+\tfrac{1}{2}\frac{A'_1}{k_2}-\frac{2a^2V}{\sqrt{2}}k_2\right\}\\&+\left\{-\tfrac{1}{2}A'_0(L_2-3)-\tfrac{1}{2}\frac{A'_1}{k_2^2}-\frac{2a^2V}{\sqrt{2}}\right\}k_2\cos v\\&+\left[-\tfrac{3}{8}A'_0(2L_2-5)-\frac{2A'_2}{k_2^4}-\frac{6a^2V}{\sqrt{2}}\right.\\&\left.+\tfrac{1}{2}\frac{\beta_1}{k_2}\left\{-\tfrac{1}{2}A'_0(L_2-4)+\tfrac{1}{2}\frac{A'_1}{k_2^2}-\frac{2a^2V}{\sqrt{2}}\right\}\right]k_2^2\cos 2v\end{aligned}\right\} . \quad (20)$$

Further, to the third order of small quantities

$$\frac{1}{\sqrt{2}}\frac{\delta\psi_2}{\delta v}=-\left\{-\tfrac{1}{2}A'_0(L-2)+\tfrac{1}{2}\frac{A'_1}{k^2}-\frac{2a^2V}{\sqrt{2}}\right\}k\sin v$$

and along the surface

$$=-\left\{-\tfrac{1}{2}\mathrm{A}'_0(\mathrm{L}_2-2)+\tfrac{1}{2}\frac{\mathrm{A}'_1}{k_2{}^2}-\frac{2a^2\mathrm{V}}{\sqrt{2}}\right\}k_2\sin v$$

$$+\text{ terms of the fourth order.}$$

Hence by (15)

$$\frac{1}{\sqrt{2}}\frac{\delta\psi_2}{\delta v}=\tfrac{1}{2}\mathrm{A}'_0\beta_1\sin v+\text{terms of the fourth order} \quad . \quad . \quad . \quad . \quad (21)$$

7. *The function* ψ_1.—The part depending on $\mathrm{A}\rho^4$ and B must be carried to the fourth order. The part depending on the A_n is similar to the expression already deduced for ψ_2, without the terms in V.

$$\tfrac{1}{8}\mathrm{A}\rho^4=\tfrac{1}{8}\mathrm{A}a^4\left(\frac{1-k^2}{1+k^2-2kc}\right)^4$$

$$=\tfrac{1}{8}\mathrm{A}a^4\{1+12k^2+42k^4+8k(1+6k^2)\cos v+20k^2(1+4k^2)\cos 2v$$

$$+40k^3\cos 3v+70k^4\cos 4v\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

$$\frac{1}{\sqrt{(\mathrm{C}-c)}}\mathrm{B}_0\mathrm{T}_0=\frac{\pi k^{-\frac{1}{2}}}{4\sqrt{(\mathrm{C}-c)}}\mathrm{B}_0\{1+\tfrac{1}{4}k^2+\tfrac{1}{64}k^4\}$$

$$=\frac{\pi\sqrt{2}}{4}\mathrm{B}_0\{1+\tfrac{1}{2}k^2+\tfrac{7}{32}k^4+k(1+\tfrac{5}{8}k^2)\cos v$$

$$+\tfrac{1}{4}k^2(3+2k^2)\cos 2v+\tfrac{5}{8}k^3\cos 3v+\tfrac{35}{64}k^4\cos 4v\}$$

$$\frac{1}{\sqrt{(\mathrm{C}-c)}}\mathrm{B}_1\mathrm{T}_1\cos v=\frac{3\pi k^{\frac{1}{2}}}{8\sqrt{(\mathrm{C}-c)}}(1-\tfrac{1}{8}k^2)\cos v$$

$$=\frac{3\pi\sqrt{2}}{8}\mathrm{B}_1k\{\tfrac{1}{2}k+\tfrac{1}{8}k^3+(1+\tfrac{1}{2}k^2)\cos v+\tfrac{1}{2}k(1+\tfrac{7}{8}k^2)\cos 2v$$

$$+\tfrac{3}{8}k^2\cos 3v+\tfrac{5}{16}k^3\cos 4v\}$$

$$\frac{1}{\sqrt{(\mathrm{C}-c)}}\mathrm{B}_2\mathrm{T}_2\cos 2v=\frac{15\pi k^{\frac{3}{2}}}{32\sqrt{(\mathrm{C}-c)}}(1-\tfrac{1}{4}k^2)\cos 2v$$

$$=\frac{15\pi\sqrt{2}}{32}\mathrm{B}_2k^2\{\tfrac{3}{8}k^2+\tfrac{1}{2}k\cos v+\cos 2v+\tfrac{1}{2}k\cos 3v+\tfrac{3}{8}k^2\cos 4v\}$$

$$\frac{1}{\sqrt{(\mathrm{C}-c)}}\mathrm{B}_3\mathrm{T}_3\cos 3v=\frac{35\pi k^{\frac{3}{2}}}{64\sqrt{(\mathrm{C}-c)}}\mathrm{B}_3\cos 3v$$

$$=\frac{35\pi\sqrt{2}}{64}\mathrm{B}_3k^3(\tfrac{1}{2}k\cos 2v+\cos 3v+\tfrac{1}{2}k\cos 4v)$$

$$\frac{1}{\sqrt{(\mathrm{C}-c)}}\mathrm{B}_4\mathrm{T}_4\cos 4v=\pi\sqrt{2}\frac{5.7.9}{2^9}k^4\cos 4v$$

Hence $\frac{\psi_1}{\pi\sqrt{2}}$ (so far as it depends on B)=

$$\left.\begin{aligned}
&\tfrac{1}{4}\mathrm{B}_0(1+\tfrac{1}{2}k^2+\tfrac{7}{32}k^4)+\tfrac{3}{16}\mathrm{B}_1k^2(1+\tfrac{1}{4}k^2)+\tfrac{45}{256}\mathrm{B}_2k^4+\tfrac{1}{8}\frac{\mathrm{A}a^4}{\pi\sqrt{2}}(1+12k^2+42k^4)\\
&+\left\{\tfrac{1}{4}\left(\mathrm{B}_0+\tfrac{3}{2}\mathrm{B}_1+\frac{4\mathrm{A}a^4}{\pi\sqrt{2}}\right)+\tfrac{1}{8}\left(\tfrac{5}{4}\mathrm{B}_0+\tfrac{3}{2}\mathrm{B}_1+\tfrac{15}{8}\mathrm{B}_2+\frac{48\mathrm{A}a^4}{\pi\sqrt{2}}\right)k^2\right\}k\cos v\\
&+\left\{\tfrac{3}{16}\left(\mathrm{B}_0+\mathrm{B}_1+\tfrac{5}{2}\mathrm{B}_2+\frac{40\mathrm{A}a^4}{3\pi\sqrt{2}}\right)+\tfrac{1}{16}\left(2\mathrm{B}_0+\tfrac{21}{8}\mathrm{B}_1+\tfrac{35}{8}\mathrm{B}_3+\frac{160\mathrm{A}a^4}{\pi\sqrt{2}}\right)k^2\right\}k^2\cos 2v\\
&+\tfrac{1}{32}\left(5\mathrm{B}_0+\tfrac{9}{2}\mathrm{B}_1+\tfrac{15}{2}\mathrm{B}_2+\tfrac{35}{2}\mathrm{B}_3+\frac{160\mathrm{A}a^4}{\pi\sqrt{2}}\right)k^3\cos 3v\\
&+\tfrac{1}{128}\left(\tfrac{35}{2}\mathrm{B}_0+15\mathrm{B}_1+\tfrac{45}{2}\mathrm{B}_2+35\mathrm{B}_3+\frac{5.7.9}{4}\mathrm{B}_4+35\times 32.\frac{\mathrm{A}a^4}{\pi\sqrt{2}}\right)k^4\cos 4v
\end{aligned}\right\} \qquad (23)$$

The next step is to substitute in this the value of k along the two surfaces, which will give the value of $\psi_1/\sqrt{2}$ along these surfaces. Suppose for the outer surface it becomes

$$p_0+p_1\cos v+p_2\cos 2v+p_3\cos 3v+p_4\cos 4v$$

The substitution $k=k_2(1+\beta_1\cos v+\beta_2\cos 2v+\ldots)$ being made it will be found that

$$\begin{aligned}
p_1=\frac{\pi}{4}&\left(\mathrm{B}_0+\tfrac{3}{2}\mathrm{B}_1+\frac{4\mathrm{A}a^4}{\pi\sqrt{2}}\right)k_2+\frac{\pi}{4}\left(\tfrac{5}{8}\mathrm{B}_0+\tfrac{3}{4}\mathrm{B}_1+\tfrac{15}{16}\mathrm{B}_2+\frac{24\mathrm{A}a^4}{\pi\sqrt{2}}\right)k_2^{\,3}\\
&+\frac{3\pi}{16}\beta_1k_2^{\,2}\left(\mathrm{B}_0+\mathrm{B}_1+\tfrac{5}{2}\mathrm{B}_2+\frac{40\mathrm{A}a^4}{3\pi\sqrt{2}}\right)+\frac{\pi}{4}\beta_1k_2^{\,2}\left(\mathrm{B}_0+\tfrac{3}{2}\mathrm{B}_1+\frac{12\mathrm{A}a^4}{\pi\sqrt{2}}\right)\\
&+\frac{\pi}{8}\beta_2k_2\left(\mathrm{B}_0+\tfrac{3}{2}\mathrm{B}_1+\frac{4\mathrm{A}a^4}{\pi\sqrt{2}}\right)
\end{aligned}$$

and the complete coefficient of cos v is found by adding the terms in A_n, viz., (15)

$$-\tfrac{1}{2}\mathrm{A}_0(\mathrm{L}_2-2)k_2+\tfrac{1}{2}\frac{\mathrm{A}_1}{k_2}+\tfrac{1}{2}\mathrm{A}_0\beta_1$$

This complete coefficient must vanish. Hence remembering that A_0 is of the second order with reference to $\mathrm{A}a^4$, it follows that

$$\mathrm{B}_0+\tfrac{3}{2}\mathrm{B}_1+\frac{4\mathrm{A}a^4}{\pi\sqrt{2}}=e_1 \quad \ldots\ldots\ldots\ldots \quad (24)$$

where e_1 is of the second order of small quantities, and can therefore be put equal to zero when multiplied by quantities of the third or higher orders.

Again, it will be found that

$$p_2=\tfrac{3}{16}\left(\mathrm{B}_0+\mathrm{B}_1+\tfrac{5}{2}\mathrm{B}_2+\frac{40\mathrm{A}a^4}{3\pi\sqrt{2}}\right)k_2^{\,2}+\tfrac{1}{16}\left(2\mathrm{B}_0+\tfrac{21}{8}\mathrm{B}_1+\tfrac{35}{8}\mathrm{B}_3+\frac{160\mathrm{A}a^4}{\pi\sqrt{2}}\right)k_2^{\,4}$$

$$+\tfrac{1}{8}\left(\mathrm{B}_0+\tfrac{3}{2}\mathrm{B}_1+\frac{4\mathrm{A}a^4}{\pi\sqrt{2}}\right)\beta_1k_2+\tfrac{3}{32}\beta_1^{\,2}k_2^{\,2}\left(\mathrm{B}_0+\mathrm{B}_1+\tfrac{5}{2}\mathrm{B}_2+\frac{40\mathrm{A}a^4}{3\pi\sqrt{2}}\right)$$

$$+\tfrac{3}{32}\beta_1k_2^{\,3}\left(\tfrac{5}{2}\mathrm{B}_0+\tfrac{9}{4}\mathrm{B}_1+\tfrac{15}{4}\mathrm{B}_2+\tfrac{35}{4}\mathrm{B}_3+\frac{80\mathrm{A}a^4}{\pi\sqrt{2}}\right)$$

$$+\tfrac{1}{8}\beta_1k_2^{\,3}\left(\tfrac{15}{8}\mathrm{B}_0+\tfrac{9}{4}\mathrm{B}_1+\tfrac{45}{16}\mathrm{B}_2+\frac{72\mathrm{A}a^4}{\pi\sqrt{2}}\right)+\tfrac{1}{8}(2\beta_2+\tfrac{1}{2}\beta_1^{\,2})k_2^{\,2}\left(\mathrm{B}_0+\tfrac{3}{2}\mathrm{B}_1+\frac{12\mathrm{A}a^4}{\pi\sqrt{2}}\right)$$

$$+\tfrac{1}{8}\beta_3k_2\left(\mathrm{B}_0+\tfrac{3}{2}\mathrm{B}_1+\frac{4\mathrm{A}a^4}{\pi\sqrt{2}}\right)$$

and the complete coefficient of cos $2v$ is found as before by adding

$$-\tfrac{1}{2}\mathrm{A}_0(\tfrac{1}{4}\beta_1^{\,2}-\beta_2)-\tfrac{3}{8}\mathrm{A}_0(\mathrm{L}_2-2)k_2^{\,2}+\tfrac{1}{4}\mathrm{A}_1+\frac{\mathrm{A}_2}{k_2^{\,2}}-\tfrac{1}{4}\beta_1\left\{\mathrm{A}_0(\mathrm{L}_2-3)k_2+\frac{\mathrm{A}_1}{k_2}\right\}$$

Remembering the remark as to the order of A_0, &c., it follows that

$$\mathrm{B}_0+\mathrm{B}_1+\tfrac{5}{2}\mathrm{B}_2+\frac{40\mathrm{A}a^4}{3\pi\sqrt{2}}=e_2\quad\ldots\ldots\ldots\ldots\quad(25)$$

where e_2 is of the second order of small quantities.

The coefficients of cos $3v$, cos $4v$ are easily found, viz.:—

$$p_3=\tfrac{1}{16}\left\{\tfrac{5}{2}\mathrm{B}_0+\tfrac{9}{4}\mathrm{B}_1+\tfrac{15}{4}\mathrm{B}_2+\tfrac{35}{4}\mathrm{B}_3+\frac{80\mathrm{A}a^4}{\pi\sqrt{2}}\right\}k_2^{\,3}=0$$

$$p_4=\tfrac{5}{828}\left\{\tfrac{7}{2}\mathrm{B}_0+3\mathrm{B}_1+\tfrac{9}{2}\mathrm{B}_2+7\mathrm{B}_3+\tfrac{63}{4}\mathrm{B}_4+\frac{224\mathrm{A}a^4}{\pi\sqrt{2}}\right\}k_2^{\,4}+\tfrac{3}{2}p_3\beta_1k_2^{\,3}=0$$

These equations give, to find the values of B to the *lowest* order.

$$\tfrac{7}{2}\mathrm{B}_0+3\mathrm{B}_1+\tfrac{9}{2}\mathrm{B}_2+7\mathrm{B}_3+\tfrac{63}{4}\mathrm{B}_4+\frac{224\mathrm{A}a^4}{\pi\sqrt{2}}=0$$

$$\tfrac{5}{2}\mathrm{B}_0+\tfrac{9}{4}\mathrm{B}_1+\tfrac{15}{4}\mathrm{B}_2+\tfrac{35}{4}\mathrm{B}_3\qquad+\frac{80\mathrm{A}a^4}{\pi\sqrt{2}}=0$$

$$\mathrm{B}_0+\mathrm{B}_1+\tfrac{5}{2}\mathrm{B}_2\qquad+\frac{40\mathrm{A}a^4}{3\pi\sqrt{2}}=0$$

$$\mathrm{B}_0+\tfrac{3}{2}\mathrm{B}_1\qquad+\frac{4\mathrm{A}a^4}{\pi\sqrt{2}}=0$$

These give the following values

$$\left.\begin{aligned} B_4 &= -\tfrac{2}{63}B_0 - \tfrac{2608}{315}\cdot\frac{Aa^4}{\pi\sqrt{2}} \\ B_3 &= -\tfrac{2}{35}B_0 - \tfrac{232}{35}\cdot\frac{Aa^4}{\pi\sqrt{2}} \\ B_2 &= -\tfrac{2}{15}B_0 - \tfrac{64}{15}\cdot\frac{Aa^4}{\pi\sqrt{2}} \\ B_1 &= -\tfrac{2}{3}B_0 - \tfrac{8}{3}\cdot\frac{Aa^4}{\pi\sqrt{2}} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (26)$$

These values of the B greatly simplify the coefficients of cos v and cos $2v$. In that of cos v occurs the expression

$$\tfrac{5}{8}B_0+\tfrac{3}{4}B_1+\tfrac{15}{16}B_2+\frac{24Aa}{\pi\sqrt{2}}$$

By means of the above equations it follows at once that its value is $18Aa^4/(\pi\sqrt{2})$. The coefficient of (cos v)/$\sqrt{2}$ now becomes

$$-\tfrac{1}{2}A_0(L_2-2)k_2+\tfrac{1}{2}\frac{A_1}{k_2}+\tfrac{1}{2}A_0\beta_1+\frac{\pi}{4}e_1k_2+\frac{9Aa_4}{2\sqrt{2}}k_2{}^3+\frac{2Aa_4}{\sqrt{2}}\beta_1k^2{}_2.$$

So also the expression

$$4B_0+\tfrac{21}{4}B_1+\tfrac{35}{4}B_3+\frac{320Aa^4}{\pi\sqrt{2}}$$

occurring in the coefficient of cos $2v$ is found to be $248Aa^4/(\pi\sqrt{2})$, and the coefficient becomes

$$-\tfrac{1}{2}A_0(\tfrac{1}{4}\beta_1{}^2-\beta_2)-\tfrac{3}{8}A_0(L_2-2)k_2{}^2+\tfrac{1}{4}A_1+\frac{A_2}{k_2{}^2}-\tfrac{1}{4}\beta_1\left\{A_0(L_2-3)k_2+\frac{A_1}{k_2}\right\}$$
$$+\frac{3\pi}{16}e_2k_2{}^2+\tfrac{31}{4}\frac{Aa^4}{\sqrt{2}}k_2{}^4+\frac{\pi}{8}\beta_1k_2e_1+\tfrac{27}{4}\frac{Aa^4}{\sqrt{2}}\beta_1k_2{}^3+(\beta_2+\tfrac{1}{4}\beta_1{}^2)k_2{}^2\frac{2Aa^4}{\sqrt{2}}.$$

Now these coefficients vanish both at the outer and inner surfaces—their values for the inner will be found by writing k_1 for k_2, $\beta_1=0$ and α_2 for β_2. They are written down later in equations B.

The pressure conditions require a knowledge of $k\delta\psi/\delta k$. So far as it depends on the B it is given by (differentiating 23)

$$\frac{1}{\pi\sqrt{2}}k\frac{\delta\psi_1}{\delta k}$$
$$=\tfrac{1}{4}(k^2+\tfrac{7}{8}k^4)B_0+\tfrac{3}{16}(2k^2+k^4)B_1+\tfrac{45}{64}B_2k^4+\frac{Aa^4}{\pi\sqrt{2}}(3k^2+21k^4)$$
$$+\left\{\tfrac{1}{4}B_0k(1+\tfrac{15}{8}k^2)+\tfrac{3}{8}B_1k(1+\tfrac{3}{2}k^2)+\tfrac{45}{64}B_2k^3+\frac{Aa^4}{\pi\sqrt{2}}(k+18k^3)\right\}\cos v$$
$$+\left\{\tfrac{1}{8}B_0k^2(3+4k^2)+\tfrac{3}{8}B_1k^2(1+\tfrac{7}{4}k^2)+\tfrac{15}{16}B_2k^2+\tfrac{35}{32}B_3k^4+\frac{5Aa^4}{\pi\sqrt{2}}(k^2+8k^4)\right\}\cos 2v$$

the coefficient of cos $3v$, cos $4v$ vanishing by what has gone before.

Substituting for k, and putting in the terms involving A_n from (20), the value along the outer surface is

$$\left.\begin{aligned}
&\frac{1}{\sqrt{2}}k\frac{\delta\psi_1}{\delta k}\\
&\quad=\tfrac{1}{2}A_0\Big\{1-(L_2-1)k_2^2\Big\}+\tfrac{1}{2}\beta_1\Big\{-\tfrac{1}{2}A_0(L_2-4)k_2+\tfrac{1}{2}\frac{A_1}{k_2}\Big\}+\frac{\pi}{4}e_1k_2^2\\
&\quad+\frac{\pi}{8}e_1\beta_1k_2+\frac{Aa^4}{\sqrt{2}}\Big(2k_2^2+\tfrac{35}{2}k_2^4+\beta_1^2k_2^2+\tfrac{81}{4}\beta_1k_2^3\Big)\\
&\quad+\Big\{-\tfrac{1}{2}A_0(L_2-3)k_2-\tfrac{1}{2}\frac{A_1}{k_2}+\frac{\pi}{4}e_1k_2+\frac{Aa^4}{\sqrt{2}}(\tfrac{27}{2}k_2^3+4\beta_1k_2^2)\Big\}\cos v\\
&\quad+\Big[-\tfrac{3}{8}A_0(2L_2-5)k_2^2-\frac{2A_2}{k_2^2}+\tfrac{1}{2}\beta_1\Big\{-\tfrac{1}{2}A_0(L_2-4)+\tfrac{1}{2}\frac{A_1}{k_2^2}\Big\}k_2\\
&\quad+(\beta_2+\tfrac{1}{4}\beta_1^2)k_2^2\frac{4Aa^4}{\sqrt{2}}+\frac{\pi}{8}e_1\beta_1k_2+\frac{3\pi}{8}e_2k_2^2+\frac{Aa^4}{\sqrt{2}}(31k_2^4+\tfrac{81}{4}\beta_1k_2^3)\Big]\cos 2v
\end{aligned}\right\}\quad . \quad (27)$$

The lowest term in $-\delta\psi_1/\delta v$ is

$$\sqrt{2}\Big\{-\tfrac{1}{2}A_0(L-2)k+\tfrac{1}{2}\frac{A_1}{k}+\frac{\pi}{4}e_1k+\tfrac{9}{2}\frac{Aa^4}{\sqrt{2}}k^3\Big\}\sin v$$

Along the outer surface this is

$$\sqrt{2}\Big\{-\tfrac{1}{2}A_0(L_2-2)k_2+\tfrac{1}{2}\frac{A_1}{k_2}+\frac{\pi}{4}e_1k_2+\tfrac{9}{2}\frac{Aa^4}{\sqrt{2}}k_2^3\Big\}\sin v$$

+ terms of the fourth order.

By Eq. (A.2) below this is

$$-\tfrac{1}{2}\beta_1\Big(A_0+\frac{4Aa^4}{\sqrt{2}}k_2^2\Big)\sin v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and is therefore of the third order.

8. *Pressure conditions.*—Along the inner surface the pressure is zero, and since it is a stream line the velocity must be constant. Denote it by U, then

$$\begin{aligned}
U^2&=\frac{1}{\rho^2}\left\{\left(\frac{\delta\psi_1}{\delta u}\right)^2+\left(\frac{\delta\psi_1}{\delta v}\right)^2\right\}\left(\frac{du}{dn}\right)^2\\
&=\frac{(C-c)^4}{a^4S^2}\left\{\left(k\frac{\delta\psi_1}{\delta k}\right)^2+\left(\frac{\delta\psi_1}{\delta v}\right)^2\right\}
\end{aligned}$$

Now along this surface $a_1=0$, and therefore $\delta\psi_1/\delta v$ is of the fourth order, whilst $k\delta\psi_1/\delta k$ is of the second. Hence it may be neglected, and

$$a^2U=-\frac{(C-c)^2}{S}k\frac{\delta\psi_1}{\delta k}$$

therefore

$$-2a^2\mathrm{U}k=(1+5k^2-4k\cos v+2k^2\cos 2v)k\frac{\delta\psi_1}{\delta k}$$

or along the surface

$$-2a^2\mathrm{U}k_1(1+\alpha_2\cos 2v)=(1+5k_1^2-4k_1\cos v+2k_1^2\cos 2v)k\frac{\delta\psi_1}{\delta k}.$$

Suppose for the moment

$$k\frac{\delta\psi_1}{\delta k}=q_0+q_1\cos v+q_2\cos 2v$$

then

$$\begin{aligned}&q_0(1+5k_1^2)-2q_1k_1=-2a^2\mathrm{U}_0k_1\\&q_1-4q_0k_1=0\\&q_2-2q_1k_1+2q_0k_1^2=-2a^2\mathrm{U}_0k_1\alpha_2.\end{aligned}$$

therefore

$$\begin{aligned}q_0(1-3k_1^2)&=-2a^2\mathrm{U}_0k_1\\q_1-4k_1q_0&=0\\q_2-6q_0k_1^2-\alpha_2q_0&=0\end{aligned}$$

These give the equations (C) below.

Before proceeding to introduce the last condition of equality of pressure on both sides of the outer boundary of the core, it will be convenient to collect here and partly discuss the equations already obtained. They are

A (no cos v in ψ_1)

$$\left.\begin{aligned}&-\tfrac{1}{2}\mathrm{A}_0(\mathrm{L}_1-2)k_1+\tfrac{1}{2}\frac{\mathrm{A}_1}{k_1}+\frac{\pi}{4}e_1k_1+\tfrac{9}{2}\frac{\mathrm{A}a^4}{\sqrt{2}}k_1^3=0\\&-\tfrac{1}{2}\mathrm{A}_0(\mathrm{L}_2-2)k_2+\tfrac{1}{2}\frac{\mathrm{A}_1}{k_2}+\frac{\pi}{4}e_1k_2+\tfrac{9}{2}\frac{\mathrm{A}a^4}{\sqrt{2}}k_2^3+\tfrac{1}{2}\beta_1\left(\mathrm{A}_0+\frac{4\mathrm{A}a^4}{\sqrt{2}}k_2^2\right)=0\end{aligned}\right\}\begin{aligned}&(1)\\&(2)\end{aligned}$$

B (no cos $2v$)

$$\left.\begin{aligned}&-\tfrac{3}{8}\mathrm{A}_0(\mathrm{L}_1-2)k_1^2+\tfrac{1}{4}\mathrm{A}_1+\frac{\mathrm{A}_2}{k_1^2}+\frac{3\pi}{16}e_2k_1^2+\tfrac{31}{4}\frac{\mathrm{A}a^4}{\sqrt{2}}k_1^4+\tfrac{1}{2}\alpha_2\left(\mathrm{A}_0+\frac{4\mathrm{A}a^4}{\sqrt{2}}k_1^2\right)=0\\&-\tfrac{3}{8}\mathrm{A}_0(\mathrm{L}_2-2)k_2^2+\tfrac{1}{4}\mathrm{A}_1+\frac{\mathrm{A}_2}{k_2^2}+\frac{3\pi}{16}e_2k_2^2+\tfrac{31}{4}\frac{\mathrm{A}a^4}{\sqrt{2}}k_2^4+\tfrac{1}{2}\beta_2\left(\mathrm{A}_0+\frac{4\mathrm{A}a^4}{\sqrt{2}}k_2^2\right)\\&-\tfrac{1}{2}\beta_1\left\{\tfrac{1}{4}\beta_1\left(\mathrm{A}\quad\frac{4\mathrm{A}a^4}{\sqrt{2}}k_2^2\right)+\tfrac{1}{2}\mathrm{A}_0(\mathrm{L}_2-3)k_2+\tfrac{1}{2}\frac{\mathrm{A}_1}{k_2}-\frac{\pi}{4}e_1k_2-\tfrac{27}{2}\frac{\mathrm{A}a^4}{\sqrt{2}}k_2^3\right\}=0\end{aligned}\right\}\begin{aligned}&(1)\\&(2)\end{aligned}$$

C (constant pressure inside)

$$\left.\begin{aligned}&\tfrac{1}{2}\mathrm{A}_0\{1-(\mathrm{L}_1-1)k_1^2\}+\frac{\pi}{4}e_1k_1^2+\frac{2\mathrm{A}a^4}{\sqrt{2}}k_1^2+\tfrac{35}{2}\frac{\mathrm{A}a^4}{\sqrt{2}}k_1^4=-\frac{2a^2\mathrm{U}k_1}{\sqrt{2}}(1+3k_1^2) &(1)\\ &-\tfrac{1}{2}\mathrm{A}_0(\mathrm{L}_1+1)k_1-\tfrac{1}{2}\frac{\mathrm{A}_1}{k_1}+\frac{\pi}{4}e_1k_1+\tfrac{11}{2}k_1^3\frac{\mathrm{A}a^4}{\sqrt{2}}=0 &(2)\\ &-\tfrac{3}{8}\mathrm{A}_0(2\mathrm{L}_1+3)k_1^2-\frac{2\mathrm{A}_2}{k_1^2}-\tfrac{1}{2}\alpha_2\left(\mathrm{A}_0-\frac{4\mathrm{A}a^4}{\sqrt{2}}k_1^2\right)+\frac{3\pi}{8}e_2k_1^2+19\frac{\mathrm{A}a^4}{\sqrt{2}}k_1^4=0 &(3)\end{aligned}\right\}$$

Values of A_0, A_1.—Subtract C (2) from A (1) and divide by k_1

$$\tfrac{3}{2}\mathrm{A}_0+\frac{\mathrm{A}_1}{k_1^2}-\frac{\mathrm{A}a^4}{\sqrt{2}}k_1^2=0$$

therefore

$$\mathrm{A}_1=-\tfrac{3}{2}\mathrm{A}_0k_1^2+\frac{\mathrm{A}a^4}{\sqrt{2}}k_1^4 \quad \ldots\ldots\ldots\ldots \quad (29)$$

Now (10)

$$\mathrm{A}_0+\mathrm{A}_1+\mathrm{A}_2=-\frac{\mu_1 a}{\pi\sqrt{2}}-\frac{4\mathrm{A}a^4}{\sqrt{2}}(k_1^2+4k_1^4)$$

Therefore to the fourth order

$$(1-\tfrac{3}{2}k_1^2)\mathrm{A}_0=-\frac{\mu_1 a}{\pi\sqrt{2}}-\frac{\mathrm{A}a^4}{\sqrt{2}}k_1^2(4+17k_1^2)$$

or

$$\mathrm{A}_0=-\frac{\mu_1 a}{\pi\sqrt{2}}(1+\tfrac{3}{2}k_1^2)-\frac{4\mathrm{A}a^4}{\sqrt{2}}k_1^2(1+\tfrac{23}{4}k_1^2)$$

or to the lowest orders respectively

$$\left.\begin{aligned}\mathrm{A}_0&=-\frac{\mu_1 a}{\pi\sqrt{2}}-\frac{4\mathrm{A}a^4}{\sqrt{2}}k_1^2\\ \mathrm{A}_1&=\tfrac{3}{2}\frac{\mu_1 a}{\pi\sqrt{2}}k_1^2+\frac{7\mathrm{A}a^4}{\sqrt{2}}k_1^4\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (29a)$$

Value of β_1.—In equations A divide (1) by k_1, (2) by k_2, and subtract

$$\tfrac{1}{2}\mathrm{A}_0(\mathrm{L}_2-\mathrm{L}_1)+\tfrac{1}{2}\mathrm{A}_1\frac{k_2^2-k_1^2}{k_1^2k_2^2}-\tfrac{9}{2}\frac{\mathrm{A}a^4}{\sqrt{2}}(k_2^2-k_1^2)-\tfrac{1}{2}\frac{\beta_1}{k_2}\left(\mathrm{A}_0+\frac{4\mathrm{A}a^4}{\sqrt{2}}k_2^2\right)=0$$

Therefore

$$\begin{aligned}\tfrac{1}{2}\frac{\beta_1}{k_2}\left(\mathrm{A}_0+\frac{4\mathrm{A}a^4}{\sqrt{2}}k_2^2\right)&=\tfrac{1}{2}\mathrm{A}_0(\mathrm{L}_2-\mathrm{L}_1)-\tfrac{3}{4}\mathrm{A}_0\left(1-\frac{k_1^2}{k_2^2}\right)-\tfrac{1}{2}\frac{\mathrm{A}a^4}{\sqrt{2}}(k_2^2-k_1^2)\left(9-\frac{k_1^2}{k_2^2}\right)\\ &=-\tfrac{1}{2}\mathrm{A}_0\left\{\log\frac{k_2}{k_1}+\tfrac{3}{2}\left(1-\frac{k_1^2}{k_2^2}\right)\right\}-\tfrac{1}{2}\frac{\mathrm{A}a^4}{\sqrt{2}}(k_2^2-k_1^2)\left(9-\frac{k_1^2}{k_2^2}\right)\end{aligned}$$

or substituting for A_0

$$\frac{\beta_1}{k_2}\left\{-\frac{\mu_1 a}{\pi\sqrt{2}}+\frac{4Aa^4}{\sqrt{2}}(k_2^2-k_1^2)\right\}=\left\{\frac{\mu_1 a}{\pi\sqrt{2}}-\frac{\mu' a}{\pi\sqrt{2}}\cdot\frac{k_1^2}{k_2^2-k_1^2}\left\{\log\frac{k_2}{k_1}+\tfrac{3}{2}\left(1-\frac{k_1^2}{k_2^2}\right)\right\}\right.$$
$$\left.+\tfrac{1}{4}\frac{\mu' a}{\pi\sqrt{2}}\left(9-\frac{k_1^2}{k_2^2}\right)\right.$$

whence

$$\frac{\beta_1}{k_2}=-\frac{\mu_1-\frac{k_1^2}{k_2^2-k_1^2}\mu'}{\mu_1+\mu'}\left\{\log\frac{k_2}{k_1}+\tfrac{3}{2}\left(1-\frac{k_1^2}{k_2^2}\right)\right\}-\tfrac{1}{4}\frac{\mu'}{\mu_1+\mu'}\left(9-\frac{k_1^2}{k_2^2}\right). \quad . \quad . \quad . \qquad (30)$$

Case (1) when there is no added circulation

$$\frac{\beta_1}{k_2}=\frac{k_1^2}{k_2^2-k_1^2}\log\frac{k_2}{k_1}+\tfrac{7}{4}\frac{k_1^2}{k_2^2}-\tfrac{9}{4}$$

Case (2) when the added circulation is very large

$$\frac{\beta_1}{k_2}=-\left\{\log\frac{k_2}{k_1}+\tfrac{3}{2}\left(1-\frac{k_1^2}{k_2^2}\right)\right\}$$

Value of e_1.—Substituting for A_1 in equation A (1)

$$-\tfrac{1}{2}A_0(L_1-\tfrac{1}{2})+\frac{\pi}{4}e_1+5\frac{Aa^4}{\sqrt{2}}k_1^2=0$$

or

$$\frac{\pi}{4}e_1=-\tfrac{1}{2}\left(\frac{\mu_1 a}{\pi\sqrt{2}}-\frac{k_1^2}{k_2^2-k_1^2}\frac{\mu' a}{\pi\sqrt{2}}\right)L_1+\frac{\mu_1 a}{4\pi\sqrt{2}}+\frac{k_1^2}{k_2^2-k_1^2}\frac{\mu' a}{\pi\sqrt{2}}$$
$$=-\frac{a}{\pi\sqrt{2}}\left\{\tfrac{1}{2}\left(\mu_1-\frac{k_1^2}{k_2^2-k_1^2}\mu'\right)\log\frac{4}{R_1}-\tfrac{1}{4}\mu_1-\frac{k_1^2}{k_2^2-k_1^2}\mu'\right\} \quad . \quad . \quad . \quad . \qquad (31)$$

Value of U_1.—Substituting in C (1), the surface velocity is given by

$$-\frac{2a^2U_1k_1}{\sqrt{2}}(1+3k_1^2)=\tfrac{1}{2}A_0(1+\tfrac{1}{2}k_1^2)+\frac{2Aa^4}{\sqrt{2}}k_1^2+\tfrac{25}{2}\frac{Aa^4}{\sqrt{2}}k_1^4$$

$$U_1=-\frac{k_1^{-1}}{a^2\sqrt{2}}\left\{\tfrac{1}{2}A_0(1-\tfrac{5}{2}k_1^2)+\frac{2Aa^4}{\sqrt{2}}k_1^2+\tfrac{13}{2}\frac{Aa^4}{\sqrt{2}}k_1^4\right\}$$
$$=\frac{k_1^{-1}}{2a^2\sqrt{2}}\left\{\frac{\mu_1 a}{\pi\sqrt{2}}(1-k_1^2)+\frac{4Aa^4}{\sqrt{2}}k_1^2(1+\tfrac{13}{4}k_1^2)-\frac{4Aa^4}{\sqrt{2}}k_1^2-13\frac{Aa^4}{\sqrt{2}}k_1^4\right\}$$
$$=\frac{\mu_1}{4\pi a k_1}(1-k_1^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (32)$$

This is the velocity along the surface of the core, not just outside it.

Equations B (1) (2) and C (3) become when the values already obtained are substituted

$$\left.\begin{aligned}
&-\tfrac{3}{8}A_0(L_1-1)k_1^2+\frac{A_2}{k_1^2}+\frac{3\pi}{16}e_2k_1^2+8\frac{Aa^4}{\sqrt{2}}k_1^4+\tfrac{1}{2}\alpha_2\left(A_0+\frac{4Aa^4}{\sqrt{2}}k_1^2\right)=0\\
&-\tfrac{3}{8}A_0(L_2-1)k_2^2+\tfrac{3}{8}A_0(k_2^2-k_1^2)+\frac{A_2}{k_2^2}+\frac{3\pi}{16}e_2k_2^2+8\frac{Aa^4}{\sqrt{2}}k_2^4-\tfrac{1}{4}.\frac{Aa^4}{\sqrt{2}}(k_2^4-k_1^4)\\
&\qquad+\tfrac{1}{2}\beta_2\left(A_0+\frac{4Aa^4}{\sqrt{2}}k_2^2\right)+\tfrac{1}{2}\frac{\beta_1}{k_2}\left\{\tfrac{1}{2}A_0(k_2^2+3k_1^2)+\frac{Aa^4}{\sqrt{2}}(9k_2^4-k_1^4)\right.\\
&\qquad\qquad\left.-\tfrac{1}{4}\beta_1k_2\left(3A_0+\frac{4Aa^4}{\sqrt{2}}k_2^2\right)\right\}=0\\
&-\tfrac{3}{8}A_0(2L_1+3)k_1^2-\frac{2A_2}{k_1^2}+\frac{3\pi}{8}e_2k_1^2+19\frac{Aa^4}{\sqrt{2}}k_1^4-\tfrac{1}{2}\alpha_2\left(A_0-\frac{4Aa^4}{\sqrt{2}}k_1^2\right)=0
\end{aligned}\right\}.\quad(33)$$

Subtracting one-half the third from the first

$$\tfrac{15}{16}A_0k_1^2+A_2\left(\frac{1}{k_1^2}+\frac{1}{k_2^2}\right)-\tfrac{3}{2}\frac{Aa^4}{\sqrt{2}}k_1^4+\tfrac{1}{4}\alpha_2\left(3A_0+\frac{4Aa^4}{\sqrt{2}}k_1^2\right)=0$$

Eliminating e_2 from the first and second

$$\left.\begin{aligned}
&-\tfrac{3}{8}A_0\left\{L_1-L_2+1-\frac{k_1^2}{k_2^2}\right\}+A_2\left(\frac{1}{k_1^4}-\frac{1}{k_2^4}\right)+\frac{8Aa^4}{\sqrt{2}}(k_1^2-k_2^2)+\tfrac{1}{4}\frac{Aa^4}{\sqrt{2}}\ \frac{k_2^4-k_1^4}{k_2^2}\\
&\qquad+\tfrac{1}{2}\frac{\alpha_2}{k_1^2}\left(A_0+\frac{4Aa^4}{\sqrt{2}}k_1^2\right)-\tfrac{1}{2}\frac{\beta_2}{k_2^2}\left(A_0+\frac{4Aa^4}{\sqrt{2}}k_2^2\right)\\
&\qquad\quad-\tfrac{1}{2}\frac{\beta_1}{k_2}\left\{\tfrac{1}{2}A_0(k_2^2+3k_1^2)+\frac{Aa^4}{\sqrt{2}}(9k_2^4-k_1^4)-\tfrac{1}{4}\beta_1k_2\left(3A_0+\frac{4Aa^4}{\sqrt{2}}k_2^2\right)\right\}=0
\end{aligned}\right\}.\quad(33a)$$

In order to determine V, the velocity of translation, and the quantities α_2, β_2, e_2, A_2, it will be necessary to consider the remaining condition, viz., of equality of pressure on both sides of the outer boundary of the core. We take the general case where the densities of the fluid in the outside and inside are different. In this case, even when there is no additional circulation in the outer fluid, the velocities on the two sides of the surface will be different. Let dashed letters refer to the outer fluid; also let p denote pressure, d density, and let Π = pressure at an infinite distance. Then

$$\frac{p'}{d'}=\text{const}-\tfrac{1}{2}\,(\text{vel})^2$$

therefore

$$p'=\Pi-\tfrac{1}{2}d'.U'^2.$$

If U_2 denote the velocity along the surface inside the core

$$p=\text{const}-\tfrac{1}{2}d.U_2^2+Ad.\psi_2$$

also

$$0=\text{const}-\tfrac{1}{2}dU_1^2+Ad.\psi_1$$

therefore

$$\begin{aligned} p&=\tfrac{1}{2}dU_1^2+Ad(\psi_2-\psi_1)-\tfrac{1}{2}d.U_2^2\\ &=\Pi-\tfrac{1}{2}d'U_2'^2 \end{aligned}$$

along the boundary. Suppose for the moment that

$$\begin{aligned} U_2^2&=l+m\cos v+n\cos 2v\\ U_2'^2&=l'+m'\cos v+n'\cos 2v \end{aligned}$$

Then the equations of condition are

$$\left.\begin{aligned} \Pi-\tfrac{1}{2}d'l'&=\tfrac{1}{2}dU_1^2+Ad(\psi_2-\psi_1)-\tfrac{1}{2}dl\\ d'm'&=dm\\ d'n'&=dn \end{aligned}\right\}\quad\ldots\ldots\quad(34)$$

Now

$$\begin{aligned} U^2&=\frac{S^2}{\rho^4}\left\{\left(\frac{\delta\psi}{\delta u}\right)^2+\left(\frac{\delta\psi}{\delta v}\right)^2\right\}\\ &=\frac{(C-c)^4}{a^4S^2}\left\{\left(k\frac{\delta\psi}{\delta k}\right)^2+\left(\frac{\delta\psi}{\delta v}\right)^2\right\} \end{aligned}$$

Further

$$\frac{(C-c)^2}{S}=\frac{(1+k^2-2kc)^2}{2k(1-k^2)}=\frac{1+5\kappa^2-4k\cos v+2k^2\cos 2v}{2k}$$

and along the surface this becomes

$$\frac{1}{2k_2}\{1+5k_2^2+\tfrac{1}{2}\beta_1^2-4k_2\cos v+(\tfrac{1}{2}\beta_1^2-\beta_2+2k_2^2)\cos 2v\}$$

Also substituting in (20) the values of A'_1, A'_2 determined from (15)

$$\begin{aligned} \frac{1}{\sqrt{2}}k_2\frac{\delta\psi_2}{\delta k}&=\tfrac{1}{2}A'_0\{1-(L_2-1)k_2^2\}-\frac{2a^2V}{\sqrt{2}}k_2^2+\tfrac{1}{2}A'_0\beta_1(k_2-\tfrac{1}{2}\beta_1)\\ &\quad+\left\{-\tfrac{1}{2}A'_0(2L_2-5)-\frac{4a^2V}{\sqrt{2}}+\tfrac{1}{2}A'_0\frac{\beta_1}{k_2}\right\}k_2\cos v\\ &\quad+\left\{-A'_0(L_2-\tfrac{19}{8})-\frac{10a^2V}{\sqrt{2}}+A'_0\frac{\beta_2}{k_2^2}-\frac{\beta_1}{k_2}\left(\tfrac{1}{2}A'_0\overline{2L_2-5}+\frac{4a^2V}{\sqrt{2}}\right)\right\}k_2^2\cos 2v \end{aligned}$$

Hence

$$\begin{aligned} \frac{(C-c)^2}{a^2S}\left(k\frac{\delta\psi_2}{\delta k}\right)&=\frac{1}{a^2k_2\sqrt{2}}\left[\tfrac{1}{2}A'_0\{1+(3L_2-4)k_2^2\}+\frac{6a^2V}{\sqrt{2}}k_2^2-\tfrac{1}{2}A'_0\beta_1k_2\right.\\ &\quad+\left\{-A'_0(L_2-\tfrac{1}{2})k_2-\frac{4a^2V}{\sqrt{2}}k_2+\tfrac{1}{2}A'_0\beta_1\right\}\cos v\\ &\quad+\left\{\tfrac{1}{2}A'_0(\tfrac{1}{2}\beta_1^2+\beta_2)+\tfrac{1}{2}A'_0(2L_2-\tfrac{13}{4})k_2^2-\frac{2a^2V}{\sqrt{2}}k_2^2\right.\\ &\quad\quad\left.\left.-\beta_1k_2\left(A'_0\overline{L_2-\tfrac{3}{2}}+\frac{4a^2V}{\sqrt{2}}\right)\right\}\cos 2v\right] \end{aligned}$$

Also (21)

$$\frac{(\mathrm{C}-c)^2}{a^2\mathrm{S}}\frac{\delta\psi}{\delta v}=\frac{1}{2\sqrt{2}}\cdot\frac{1}{k_2a^2}\mathrm{A}'_0\beta_1\sin v$$

which in the square gives

$$\frac{1}{16a^4k_2{}^2}\mathrm{A}'_0{}^2\beta_1{}^2(1-\cos 2v)$$

For the first approximation we shall keep as far as $\cos v$. In this case

$$\frac{(\mathrm{C}-c)^2}{a^2\mathrm{S}}k\frac{\delta\psi}{\delta k}=\frac{1}{a^2k_2\sqrt{2}}\left[\tfrac{1}{2}\mathrm{A}'_0-\left\{\mathrm{A}'_0(\mathrm{L}_2-\tfrac{1}{2})+\frac{4a^2\mathrm{V}}{\sqrt{2}}-\tfrac{1}{2}\mathrm{A}'_0\frac{\beta_1}{k_2}\right\}k_2\cos v\right]$$

and in the square

$$\frac{1}{2a^4k_2{}^2}\left[\tfrac{1}{4}\mathrm{A}'_0{}^2-\mathrm{A}'_0\left\{\mathrm{A}'_0(\mathrm{L}_2-\tfrac{1}{2})+\frac{4a^2\mathrm{V}}{\sqrt{2}}-\tfrac{1}{2}\mathrm{A}'_0\frac{\beta_1}{k_2}\right\}k_2\cos v\right]$$

or, to lowest orders

$$l'=\frac{1}{8a^4k_2{}^2}\mathrm{A}'_0{}^2$$

$$m'=-\frac{1}{2a^4k_2}\left\{\mathrm{A}'_0(\mathrm{L}_2-\tfrac{1}{2})+\frac{4a^2\mathrm{V}}{\sqrt{2}}-\tfrac{1}{2}\mathrm{A}'_0\frac{\beta_1}{k_2}\right\}\mathrm{A}'_0$$

So, to the first three orders of k, we get from (27)

$$\frac{1}{\sqrt{2}}k_2\frac{\delta\psi_1}{\delta k}=\tfrac{1}{2}\mathrm{A}_0+\frac{2\mathrm{A}a^4}{\sqrt{2}}k_2{}^2+\left\{-\tfrac{1}{2}\mathrm{A}_0(\mathrm{L}_2-3)-\frac{\mathrm{A}_1}{2k_2{}^2}+\frac{\pi}{4}e_1+\frac{\mathrm{A}a^4}{\sqrt{2}}(\tfrac{27}{2}k_2{}^2+4\beta_1k_2)\right\}k_2\cos v$$

Substituting for e_1 and A_1 from (A.2) and (29)

$$\frac{1}{\sqrt{2}}k_2\frac{\delta\psi_1}{\delta k}=\tfrac{1}{2}\left(\mathrm{A}_0+\frac{4\mathrm{A}a^4}{\sqrt{2}}k_2{}^2\right)$$
$$+\left\{\tfrac{1}{2}\mathrm{A}_0\left(1+3\frac{k_1{}^2}{k_2{}^2}\right)+\frac{\mathrm{A}a^4}{\sqrt{2}}\left(9-\frac{k_1{}^4}{k_2{}^4}\right)k_2{}^2-\tfrac{1}{2}\frac{\beta_1}{k_2}\left(\mathrm{A}_0-\frac{4\mathrm{A}a^4}{\sqrt{2}}k_2{}^2\right)\right\}k_2\cos v$$

whence

$$\frac{(\mathrm{C}-c)^2}{a^2\mathrm{S}}k_2\frac{\delta\psi_1}{\delta k}=\frac{1}{a^2k_2\sqrt{2}}\left[\tfrac{1}{2}\left(\mathrm{A}_0+\frac{4\mathrm{A}a^4}{\sqrt{2}}k_2{}^2\right)\right.$$
$$\left.+\left\{-\tfrac{3}{2}\mathrm{A}_0\left(1-\frac{k_1{}^2}{k_2{}^2}\right)+\frac{\mathrm{A}a^4}{\sqrt{2}}\left(1-\frac{k_1{}^4}{k_2{}^4}\right)k_2{}^2-\tfrac{1}{2}\frac{\beta_1}{k_2}\left(\mathrm{A}_0-\frac{4\mathrm{A}a^4}{\sqrt{2}}k_2{}^2\right)\right\}k_2\cos v\right]$$

and the square is

$$=\frac{1}{2a^4k_2{}^2}\left[\tfrac{1}{4}\left(\mathrm{A}_0+\frac{4\mathrm{A}a^4}{\sqrt{2}}k_2{}^2\right)^2\right.$$
$$\left.+\left(\mathrm{A}_0+\frac{4\mathrm{A}a^4}{\sqrt{2}}k_2{}^2\right)\left\{-\tfrac{3}{2}\mathrm{A}_0\left(1-\frac{k_1{}^2}{k_2{}^2}\right)+\frac{\mathrm{A}a^4}{\sqrt{2}}\left(1-\frac{k_1{}^4}{k_2{}^4}\right)k_2{}^2-\tfrac{1}{2}\frac{\beta_1}{k_2}\left(\mathrm{A}_0-\frac{4\mathrm{A}a^4}{\sqrt{2}}k_2{}^2\right)\right\}k_2\cos v\right]$$

Hence, applying the conditions (34)

$$\Pi-\frac{d'}{16a^4k_2^{\,2}}\mathrm{A}_0'^{\,2}=\tfrac{1}{2}d.\mathrm{U}_1^{\,2}+\mathrm{A}d(\psi_2-\psi_1)-\frac{d}{16a^4k_2^{\,2}}\left(\mathrm{A}_0+\frac{4\mathrm{A}a^4}{\sqrt{2}}k_2^{\,2}\right)^2$$

$$-\frac{d'}{2a^4k_2}\left\{\mathrm{A}'_0(\mathrm{L}_2-\tfrac{1}{2})+\frac{4a^2\mathrm{V}}{\sqrt{2}}-\tfrac{1}{2}\mathrm{A}'_0\frac{\beta_1}{k_2}\right\}\mathrm{A}'_0$$

$$=\frac{d}{2a^4k_2}\left(\mathrm{A}_0+\frac{4\mathrm{A}a^4}{\sqrt{2}}k_2^{\,2}\right)\left\{-\tfrac{3}{2}\mathrm{A}_0\left(1-\frac{k_1^{\,2}}{k_2^{\,2}}\right)+\frac{\mathrm{A}a^4}{\sqrt{2}}\left(1-\frac{k_1^{\,4}}{k_2^{\,4}}\right)k_2^{\,2}-\tfrac{1}{2}\frac{\beta_1}{k_2}\left(\mathrm{A}_0-\frac{4\mathrm{A}a^4}{\sqrt{2}}k_2^{\,2}\right)\right\}$$

Now (29*a*) to the lowest order

$$\mathrm{A}_0+\frac{4\mathrm{A}a^4}{\sqrt{2}}k_2^{\,2}=-\frac{\mu_1 a}{\pi\sqrt{2}}+\frac{4\mathrm{A}a^4}{\sqrt{2}}(k_2^{\,2}-k_1^{\,2})$$

$$=-(\mu_1+\mu')\frac{a}{\pi\sqrt{2}}$$

and

$$\mathrm{A}'_0=-\frac{\mu_2 a}{\pi\sqrt{2}} \quad \text{from (18)}$$

$$\mathrm{U}_1=\frac{\mu_1}{4\pi a k_1} \quad \text{from (32)}$$

$$\psi_2-\psi_1=\mathrm{A}a^4(k_2^{\,2}-k_1^{\,2})-\frac{\sqrt{2}}{2}\mathrm{A}_0(\mathrm{L}_2-\mathrm{L}_1) \quad \text{from (15, 23)}$$

$$=-\frac{\mu' a}{4\pi}-\frac{1}{\sqrt{2}}\mathrm{A}_0(\mathrm{L}_2-\mathrm{L}_1)$$

Substituting, the above equations become

$$\left.\begin{aligned}&\Pi-\frac{1}{32\pi^2a^2k_2^{\,2}}\{\mu_2^{\,2}d'-(\mu_1+\mu')^2d\}=\frac{\mu_1^{\,2}}{32\pi^2a^2k_1^{\,2}}d-\mathrm{A}d\left\{\frac{\mu' a}{4\pi}+\frac{1}{\sqrt{2}}\mathrm{A}_0(\mathrm{L}_2-\mathrm{L}_1)\right\}\\ &\frac{\mu_2 a d'}{2\pi^2}\left\{-\mu_2 a(\mathrm{L}_2-\tfrac{1}{2})+4a^2\pi\mathrm{V}+\tfrac{1}{2}\mu_2 a\frac{\beta_1}{k_2}\right\}\\ &\qquad=-\frac{(\mu_1+\mu')ad}{2\pi^2}\left\{-\tfrac{3}{2}\pi\sqrt{2}\mathrm{A}_0\left(1-\frac{k_1^{\,2}}{k_2^{\,2}}\right)+\pi\mathrm{A}a^4\left(1-\frac{k_1^{\,4}}{k_2^{\,4}}\right)k_2^{\,2}\right.\\ &\qquad\qquad\left.-\frac{\pi\sqrt{2}}{2}\frac{\beta_1}{k_2}\left(\mathrm{A}_0-\frac{4\mathrm{A}a^4}{\sqrt{2}}k_2^{\,2}\right)\right\}\end{aligned}\right\} \quad (35)$$

The latter is

$$\tfrac{1}{2}\mu_2^{\,2}d'\left(\frac{\beta_1}{k_2}-2\mathrm{L}_2+1\right)a+4\mu_2 a^2\pi d'\mathrm{V}$$

$$=-(\mu_1+\mu')d\left\{\frac{3a}{2}\left(\mu_1-\frac{k_1^{\,2}}{k_2^{\,2}-k_1^{\,2}}\mu'\right)\frac{k_2^{\,2}-k_1^{\,2}}{k^2_{\,2}}-\tfrac{1}{4}\mu' a\left(1+\frac{k_1^{\,2}}{k_2^{\,2}}\right)+\frac{a}{2}\frac{\beta_1}{k_2}\left(\mu_1-\frac{k_2^{\,2}+k_1^{\,2}}{k_2^{\,2}-k_1^{\,2}}\mu'\right)\right\}$$

therefore

$$\left.\begin{aligned}4\pi a d'\mu_2 \mathrm{V}=&-\tfrac{1}{2}\mu_2{}^2 d'\left(\frac{\beta_1}{k_2}-2\mathrm{L}_2+1\right)\\&-(\mu_1+\mu')d\left\{\frac{3}{2k_2{}^2}(\mu_1\overline{k_2{}^2-k_1{}^2}-\mu' k_1{}^2)-\tfrac{1}{4}\mu'\frac{k_2{}^2+k_1{}^2}{k_2{}^2}\right\}\\&-\tfrac{1}{2}\left(\mu_1-\frac{k_2{}^2+k_1{}^2}{k_2{}^2-k_1{}^2}\mu'\right)d.(\mu_1+\mu')\frac{\beta_1}{k_2}\end{aligned}\right\} \quad . \quad . \quad (36)$$

This gives V, the velocity of translation.

The first equation is

$$\Pi=\frac{1}{32\pi^2a^2k_2{}^2}\{\mu_2{}^2d'-(\mu_1+\mu')^2d\}+\frac{\mu_1{}^2}{32\pi^2a^2k_1{}^2}d$$
$$+\frac{1}{16\pi^2a^2}\left\{\mu'-2\left(\mu_1-\frac{k_1{}^2}{k_2{}^2-k_1{}^2}\mu'\right)(\mathrm{L}_2-\mathrm{L}_1)\right\}\frac{\mu' d}{k_2{}^2-k_1{}^2}$$

or

$$32\pi^2a^2\Pi=\{\mu_2{}^2d'-(\mu_1+\mu')^2d\}\frac{1}{k_2{}^2}+\frac{\mu_1{}^2d}{k_1{}^2}+2\left\{\mu'-2\left(\mu_1-\frac{k_1{}^2}{k_2{}^2-k_1{}^2}\mu'\right)(\mathrm{L}_2-\mathrm{L}_1)\right\}\frac{\mu' d}{k_2{}^2-k_1{}^2}$$

Another equation between a, k_1, k_2 is found from the fact that the volume of the core is constant, *i.e.*, (7)

$$8\pi^2a^3(k_2{}^2-k_1{}^2)=\text{ volume }=m\text{ say}$$

therefore

$$\frac{4m\Pi}{a}=\{\mu_2{}^2d'-(\mu_1+\mu')^2d\}\left(1-\frac{k_1{}^2}{k_2{}^2}\right)+\mu_1{}^2d.\frac{1-\dfrac{k_1{}^2}{k_2{}^2}}{\dfrac{k_1{}^2}{k_2{}^2}}+2\left\{\mu'-2\left(\mu_1-\frac{\dfrac{k_1{}^2}{k_2{}^2}}{1-\dfrac{k_1{}^2}{k_2{}^2}}\mu'\right)(\mathrm{L}_2-\mathrm{L}_1)\right\}\mu' d.$$

Write

$$k_1{}^2/k_2{}^2=x$$

Then

$$\frac{4m\Pi}{a}=\{\mu_2{}^2d'-(\mu_1+\mu')^2d\}(1-x)+\mu_1{}^2d\left(\frac{1}{x}-1\right)+2\mu' d\left\{\mu'-\left(\mu_1-\frac{\mu' x}{1-x}\right)\log x\right\} \quad (37)$$

To find α_2, β_2, A_2, e_2, we must take account of one order higher and the terms in cos $2v$. This gives another equation, which, with the previous ones ($33a$), will be sufficient to determine them. As we do not require them for our present purposes we shall postpone their consideration to another occasion.

It remains to discuss the results already obtained, viz., the three expressions, which give β_1, V, and the relation between a and the k's (or R and r_1, r_2).

The formulæ are, writing x for $k_1{}^2/k_2{}^2$

$$(\mu_1+\mu')\ \frac{\beta}{k_2}=-\tfrac{1}{2}\mu_1\left\{\log\frac{1}{x}+3(1-x)\right\}-\tfrac{1}{2}\mu'\left\{\tfrac{9}{2}-\tfrac{7}{2}x-\frac{x}{1-x}\log\frac{1}{x}\right\} \quad . \quad . \quad (30)$$

$$\begin{aligned}4\pi a\mu_2 d'\mathrm{V}&=\tfrac{1}{2}\mu_2^2 d'\left\{-\frac{\beta_1}{k_2}+2\mathrm{L}_2-1\right\}\\ &-\tfrac{1}{2}(\mu_1+\mu')d\left\{3(\mu_1\overline{1-x}-\mu'x)-\tfrac{1}{2}\mu'(1+x)-\left(\mu_1-\frac{1+x}{1-x}\mu'\right)\frac{\beta_1}{k_2}\right\}\end{aligned} \quad . \quad . \quad (36)$$

$$\begin{aligned}\frac{4m\Pi}{a}&=\{\mu_2^2 d'-(\mu_1+\mu')^2 d\}(1-x)+\mu_1^2 d\left(\frac{1}{x}-1\right)\\ &+2\mu'\left\{\mu'+\left(\mu_1-\frac{x}{1-x}\mu'\right)\log\frac{1}{x}\right\}d\end{aligned} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

Discussion of results.

9. (β_1). The expression for β_1 gives the relative position of the internal hollow to the outer boundary of the core. The centre of the cross-section of the outer boundary is outside that of the inner at a distance (by 5)

$$=2a(k_2^2-k_1^2+\beta_1 k_2)=2ak_2^2\left(1-x+\frac{\beta_1}{k_2}\right)$$

This is, as we shall see, in general negative. Hence the centre of the inner lies outside that of the outer, at a distance whose ratio (y) to the radius ($2ak_2$) of the outer is

$$y=-k_2\left(1-x+\frac{\beta_1}{k_2}\right)$$

Substituting for β_1/k_2

$$\begin{aligned}(\mu_1+\mu')y&=k_2\left[\tfrac{1}{2}\mu_1\left\{\log\frac{1}{x}+3(1-x)\right\}+\tfrac{1}{2}\mu'\left\{\tfrac{9}{2}-\tfrac{7}{2}x-\frac{x}{1-x}\log\frac{1}{x}\right\}-(\mu_1+\mu')(1-x)\right]\\ &=k_2\left\{\tfrac{1}{2}\mu_1\left(1-x+\log\frac{1}{x}\right)+\tfrac{1}{2}\mu'\left(\tfrac{5}{2}-\tfrac{3}{2}x-\frac{x}{1-x}\log\frac{1}{x}\right)\right\}\\ &=\mu_1 y_1+\mu' y' \text{ (say)}\end{aligned} \quad . \quad . \quad . \quad . \quad . \quad (38)$$

where y_1, y' are the values of y when μ_1 is very large, and zero respectively. If therefore we know the points at which the centres lie when the added circulation is very large, and when it is zero, the actual position is the centre of gravity of two masses proportional to the circulations, placed at the corresponding centres. It is only necessary therefore to discuss the values of y_1, y' separately.

When there is no added circulation

$$y'=\tfrac{1}{2}\left(\frac{5-3x}{2}-\frac{x}{1-x}\log\frac{1}{x}\right)k_2$$

When there is no hollow, $x=0$, $y'=\frac{5}{4}k_2$. This is the point at which, if the pressure be diminished, or the ring be increased large enough, the hollow will begin to form.

Again, when k_2 is uniform,

$$\frac{dy'}{dx}=-\tfrac{3}{4}+\frac{1}{1-x}-\frac{1}{(1-x)^2}\log\frac{1}{x}$$

which is always negative, or y' continually decreases as x increases; while when $x=1$, $y'=0$, so that y' is always positive.

Hence, keeping the outside boundary the same, if the mass of the core be gradually diminished (or the hollow increased) the centre of the hollow moves in, from a point $(\tfrac{5}{4}k_2)^{\text{th}}$ of the radius, towards the centre of the outer boundary, coinciding with this last in the limit.

When the added circulation is very large

$$y_1=\tfrac{1}{2}\left(1-x+\log\frac{1}{x}\right)k_2$$

This is always positive. When $x=1$, or there is no rotational motion, $y_1=0$, as x decreases, y increases, and the hollow moves outwards to the boundary. From the formula itself we should gather that when the radius of the hollow is decreased to a certain small amount, it will be in contact with the outer boundary, and if this were possible the hollow would slip out of the inside of the core into the fluid, and there would be two rings formed, one with a hollow only and circulation round it, and another with rotational core and no added circulation. But this cannot be asserted until we have learnt the connexion between k_2 and a, from equation (37), for in the actual case it will be impossible to reduce k_1 below a certain limit to be determined. It is easy to see that $y_1>y'$.

An idea of the magnitudes of the quantities involved can best be obtained from a numerical example. Take for instance the case of a ring 10 cm. radius, and radius of cross-section 1 cm. Then $k_2=\tfrac{1}{20}$. Take the three cases where the radii of the cross-section of the hollow are respectively $\tfrac{1}{4}$, $\tfrac{1}{2}$, $\tfrac{3}{4}$, we find

$$x=\tfrac{1}{16}\qquad y'=\cdot055\qquad y_1=\cdot093$$
$$x=\tfrac{1}{4}\qquad y'=\cdot041\qquad y_1=\cdot053$$
$$x=\tfrac{9}{16}\qquad y'=\cdot023\qquad y_1=\cdot026$$

The limit when $y_1=1$ is found from

$$\log\frac{1}{x}=39+x$$

or writing

$$x=e^{-39-\xi}$$

$$\xi=e^{-39}=10^{-16}\times\cdot115$$

i.e.

$$x=10^{-16}\times\cdot115$$

a sufficiently small quantity.

10. *Relation between a, k_1, k_2.*—When there is no hollow the volume of the core remains constant. In other words, $a^3k^2_2$ is constant. When a hollow exists the relation is found from the equation

$$\frac{4m\Pi}{a}=\{\mu_2^2d'-(\mu_1+\mu')^2d\}(1-x)+\mu_1^2d\left(\frac{1}{x}-1\right)+2\mu'\left\{\mu'+\left(\mu_1-\frac{x}{1-x}\mu'\right)\log\frac{1}{x}\right\}d \quad (37)$$

combined with that of constant volume, viz.:

$$m=8\pi^2a^3(k_2^2-k_1^2)$$

The first gives the ratio k_1/k_2, the second then gives k_1 or k_2 in terms of a, and a is determined by the energy of the motion, which is considered below.

In the case where there is no core,

$$\mu_1=\mu'=0,\ m=0,\ k_2=k_1$$

and

$$\Pi=\frac{\mu^2_2d'}{32\pi^2a^2k^2_2}$$

or

$$2ak_2=\frac{\mu_2}{2\pi}\sqrt{\frac{d'}{2\Pi}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

or, the sectional radius of the hollow remains constant. This agrees with the result in [I. § 9].

It is clear that when there is an added circulation this radius cannot become very small. For we may write the equation

$$\mu_1d\left\{\frac{\mu_1}{x}+2\mu'\log\frac{1}{x}\right\}=\frac{4m\Pi}{a}-\{\mu_2^2d'-(\mu_1+\mu')^2d\}(1-x)+\mu_1^2d$$
$$-2\mu'^2d+2\mu'^2\frac{x}{1-x}\log\left(\frac{1}{x}\right)d$$

Now as x decreases from 1 to 0, the right hand side continually decreases. Hence the greatest value is when $x=1$

Therefore

$$\mu_1d\left(\frac{\mu_1}{x}+2\mu'\log\frac{1}{x}\right)<\frac{4m\Pi}{a}+\mu_1^2d$$
$$>\frac{4m\Pi}{a}-\mu_2^2d'+\{(\mu_1+\mu')^2+\mu_1^2-2\mu'^2\}d$$

Hence

$$x>\frac{\mu_1^2d+2\mu_1\mu'xd\log\frac{1}{x}}{\mu_1^2d+\frac{4m\Pi}{a}}>\frac{\mu_1^2d}{\mu_1^2d+\frac{4m\Pi}{a}}$$

This, therefore, is an inferior limit to x.

Now we saw above that $y_1=\frac{1}{2}\left(1-x+\log\frac{1}{x}\right)k_2$.

In the case where x is very small, the value of y depending on the added circulation is

$$\frac{\mu_1 y_1}{\mu_1+\mu'}=\frac{\mu_1 k_2}{2(\mu_1+\mu')}\log\frac{1}{x}$$

$$\text{and } x>\frac{\mu_1^2 d}{\mu_1^2 d+\frac{4m\Pi}{a}}$$

therefore

$$\frac{\mu_1 y_1}{\mu_1+\mu'}<\frac{\mu_1 k_2}{2(\mu_1+\mu')}\log\left(1+\frac{4m\Pi}{\mu_1^2 da}\right).$$

When $\mu_1^2 da$ is large compared with $4m\Pi$

$$y<\frac{2m\Pi k_2}{\mu_1^2 da}.$$

Hence, the centre of the hollow never approaches near the outer boundary.

Two or three cases further may be usefully considered. When there are no added circulations, $\mu_1=0$, $\mu_2=\mu'$, and

$$\frac{4m\Pi}{a}=\mu^2\left\{(1-x)d'+(1+x)d-\frac{2xd}{1-x}\log\frac{1}{x}\right\} \quad . \quad . \quad . \quad . \quad . \quad (40)$$

This gives the condition when a continuous core begins to develop a hollow. In this case $x=0$, and

$$\frac{4m\Pi}{a}=\mu^2(d+d')$$

or

$$a=\frac{4m\Pi}{\mu^2(d+d')} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

Now a depends on the energy and increases with it. Hence as the energy increases a point is at last reached at which a continuous ring begins to develop a hollow. The sectional radius r_1 is then given by

$$8\pi^2\Pi r_1^2=\mu_2^2(d+d')$$

If a_1 be the value of a when a hollow is just formed, then in general

$$\frac{a_1}{a}(d+d')=(1-x)d'+(1+x)d-\frac{2xd}{1-x}\log 1/x,$$

or if $d'/d=\rho$

$$\frac{a_1}{a}(1+\rho)=(1-x)\rho+1+x-\frac{2x}{1-x}\log 1/x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

So also if r_0 be the radius of the hollow when $a=\infty$ and r for any other value of x

$$\frac{r^2}{r_0^2}\mu^2{}_2\rho=\mu^2{}_2\rho-(\mu_1+\mu')^2+\frac{\mu_1{}^2}{x}+2\mu'^2\frac{1-x+x\log x}{(1-x)^2}+2\mu_1\mu'\frac{\log 1/x}{1-x} \quad . \quad . \quad (43)$$

Where also $8\pi^2\Pi r^2{}_0=\mu^2{}_2d'$.

Hence $r^2{}_0(1+\rho)=r_1{}^2\rho$, or when the hollow first forms the sectional radius is $\sqrt{\left(\frac{1+\rho}{\rho}\right)}$ its ultimate value.

It will be important to know how the section of the core alters as the aperture increases. If r be the radius of the section, $r=2ak_2$. Also $m=8\pi^2a^3(k^2{}_2-k^2{}_1)$ $=2\pi^2ar^2(1-x)$.

Hence

$$\frac{da}{a}+2\frac{dr}{r}-\frac{dx}{1-x}=0.$$

Differentiating (37) and substituting for $dx/(1-x)$ it will be found that

$$\left[\mu_1{}^2\frac{(1-x)^2}{x^2}+2\mu_1\mu'\frac{1-x}{x}-4\mu'^2+2\mu'\left(\frac{1+x}{1-x}\mu'-\mu_1\right)\log\frac{1}{x}\right]\frac{da}{a}$$

$$=-2\left[\left\{\mu_2{}^2\rho-(\mu_1+\mu')^2\right\}(1-x)+\mu_1{}^2\frac{1-x}{x^2}+2\mu'\left(\mu_1\frac{1-x}{x}-\mu'\right)+\frac{2\mu'^2}{1-x}\log\frac{1}{x}\right]\frac{dr}{r}.$$

Here the left hand factor is always positive, and the right hand factor is clearly positive when x is small whatever the relations between the circulations and densities. Hence *the section always decreases as the aperture increases, when the interior hollow is small compared with the outside.*

When x is nearly $=1=1-z$ say, the equation becomes

$$z^2(\mu_1{}^2+\mu_1\mu'+\tfrac{1}{3}\mu'^2)\frac{da}{a}=-2z\left\{\mu_2{}^2\frac{d'}{d}+2(\mu_1{}^2-\mu_1\mu'+\tfrac{1}{3}\mu'^2)z\right\}\frac{dr}{r}$$

Hence, when z is small, so also is dr/da, and it vanishes with z. In other words, *when the hollow is large compared with the outer boundary, the section of the ring decreases very slowly as the aperture increases.* When there is no core at all we have seen that it remains absolutely constant. In all cases where there is a hollow the section decreases more slowly than if the core were continuous.

It remains to see how the radius of the hollow itself changes with increasing aperture. Here $r'^2=xr^2$. Hence

$$2\frac{dr'}{r}=\frac{dx}{x}+2\frac{dr}{r}$$

$$=\frac{1-x}{x}\left(\frac{da}{a}+\frac{2dr}{r}\right)+\frac{2dr}{r}$$

$$=\frac{1}{x}\left\{(1-x)\frac{da}{a}+2\frac{dr}{r}\right\}$$

Substituting for dr/r

$$2\frac{dr'}{r'}=\frac{1}{x}\frac{\{\mu_2{}^2\rho-(\mu_1+\mu')^2\}(1-x)^2-2\mu_1\mu'(1-x)+2\mu'^2(1+x)+2\mu'\left(\mu_1-\frac{2x}{1-x}\mu'\right)\log\frac{1}{x}}{\mathrm{D}}\quad\frac{da}{a}$$

Where D is the coefficient of $-2dr/r$ in the equation found above.

The maximum radius of the hollow will be when x has the value given by

$$\{\mu_2{}^2\rho-(\mu_1+\mu')^2\}(1-x)^2-2\mu_1\mu'(1-x)+2\mu'^2(1+x)+2\mu'\left(\mu_1-\frac{2x}{1-x}\mu'\right)\log\frac{1}{x}=0$$

This is satisfied by $x=1$.

It is easy to show that the expression on the left increases as x decreases, and therefore that dr/da is always positive. Hence

The radius of the hollow continually increases, as the aperture grows, up to coincidence with the outer boundary.

11. *Velocity of translation.* The velocity is given by the formula in (36). If the value of β_1/k_2 be substituted in the second term it becomes

$$4\pi a\mu_2 d'\mathrm{V}=\tfrac{1}{2}\mu_2{}^2d'\left(2\mathrm{L}_2-1-\frac{\beta_1}{k_2}\right)+\tfrac{1}{4}\left(\mu_1-\frac{x}{1-x}\mu'\right)\left(\mu_1-\frac{1+x}{1-x}\mu'\right)d\log\frac{1}{x}$$
$$-\tfrac{1}{4}d\left\{3(1-x)\mu_1{}^2+\tfrac{1}{2}(7-13x)\mu_1\mu'+\frac{7x^2-10x+7}{2(1-x)}\mu'^2\right\}\quad.\quad.\quad.\quad.\quad(44)$$

Case I. Where there is no core $d=0$, $\mu_1=\mu'=0$, $x=1$, $\beta_1=0$.

$$\mathrm{V}=\frac{\mu_2}{8\pi a}(2\mathrm{L}-1)\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad(45)$$

which agrees with the result obtained in [I. 21].

Case II. Continuous core. $\mu_1=0$, $k_1=0$, $x=0$, $\beta_1/k_2=-\frac{9}{4}$

$$4\pi a\mu_2 d'\mathrm{V}=\tfrac{1}{2}\mu_2{}^2d'\left(2\mathrm{L}+\frac{5}{4}\right)-\frac{7}{8}\mu'^2d\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad(46)$$

In the simplest case where there is no added circulation at all $\mu_2=\mu'$.

$$V=\frac{\mu'}{8\pi a}\left(2L+\frac{5}{4}-\frac{7}{4}\frac{d}{d'}\right)$$
$$=\frac{\mu'}{4\pi a}\left(L-\tfrac{1}{4}\right) \text{ when } d=d'.$$

In general, when x is small, the most important term is that involving $\log\frac{1}{x}$. It is given by

$$4\pi a\mu_2 d'V=\tfrac{1}{2}\mu_2^2 d'\left\{2L_2+\frac{\mu_1-\frac{x}{1-x}\mu'}{\mu_1+\mu'}(L_1-L_2)\right]+\tfrac{1}{2}\left(\mu_1-\frac{x}{1-x}\mu'\right)\left(\mu_1-\frac{1+x}{1-x}\mu'\right)(L_1-L_2)d$$

$$8\pi a\mu_2 V=\mu_2^2\left\{\left(1+\frac{\mu'}{\mu_1+\mu'}\frac{1}{1-x}\right)L_2+\left(1-\frac{\mu'}{\mu_1+\mu'}\frac{1}{1-x}\right)L_1\right]$$
$$+(\mu_1+\mu')^2\left(1-\frac{\mu'}{\mu_1+\mu'}\frac{1}{1-x}\right)\left(1-\frac{2\mu'}{\mu_1+\mu'}\cdot\frac{1}{1-x}\right)\frac{d}{d'}\log\frac{k_2}{k_1}. \quad . \quad (47)$$

When there is no added circulation at all, and $d=d'$

$$V=\frac{\mu}{8\pi a}\left\{L_2+L_1-\frac{1-2x-x^2}{(1-x)^2}(L_1-L_2)\Big)\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

As an example of the use of the formulæ let us find the alteration in V, in this simplest case, when the radius of the ring is doubled, supposing originally that the radius of the ring $=10$ cm., outside radius $=1$ cm., and inside $=\frac{1}{2}$ cm.

Since the volume is unaltered, if κ_2, κ_1 be the new values of k_2, k_1

$$8(\kappa_2^2-\kappa_1^2)=k_2^2-k_1^2=\tfrac{1}{400}-\tfrac{1}{1600}=\tfrac{3}{1600}$$

also since

$$\frac{4m\Pi}{a}=2\mu^2\left(1-\frac{x}{1-x}\log\frac{1}{x}\right)d$$

$$1-2\frac{\kappa_1^2}{\kappa_2^2-\kappa_1^2}\log\frac{\kappa_2}{\kappa_1}=\tfrac{1}{2}\left(1-2\frac{k_1^2}{k_2^2-k_1^2}\log\frac{k_2}{k_1}\right)$$

or

$$\frac{1}{z-1}\log_e z=\tfrac{1}{2}+\tfrac{1}{3}\log_e z$$

where

$$z=(\kappa_2/\kappa_1)^2$$

therefore

$$\log_{10} z=\cdot 731(z-1)\log_{10} e$$
$$=\cdot 3174(z-1)$$

whence

$$z=1{\cdot}85$$

But

$$\kappa_1^2(z-1)=\tfrac{3}{12800}$$

and

$$\kappa_1^2=\frac{3}{12800\times{\cdot}85}$$

Now

$$\frac{V'}{V}=\tfrac{1}{2}\frac{L'_2-\tfrac{1}{2}\dfrac{1}{(z-1)^2}\log z-\tfrac{1}{4}\dfrac{z+1}{(z-1)}}{L_2-\tfrac{1}{2}\dfrac{x^2}{(1-x)^2}\log 1/x-\tfrac{1}{4}\dfrac{1+x}{1-x}}$$

$$={\cdot}594.$$

If the pressure had been so great that there was no hollow at all, then

$$8\kappa^2=k^2=\tfrac{1}{400}$$

and

$$\frac{V'}{V}=\tfrac{1}{2}\frac{L'_2-\tfrac{1}{4}}{L_2-\tfrac{1}{4}}={\cdot}625.$$

These cases are however not comparable, their masses being so different.

12. *Energy.*—The energy will be composed of two parts, external and internal. Denote them by E_2, E_1 respectively, and let

$$E=E_1+E_2$$

We need only the lowest order of terms in E_1. We can easily, therefore, obtain it by supposing the core to be bounded by concentric circles, and the velocity at any point to be, V in the direction of translation, and U perpendicular to the radius to the centre of the cross-section. The energy will not contain terms in V U. Hence, R being the radius of the axis of the ring,

$$E_1=\tfrac{1}{2}.2\pi Rd.\int_{r_1}^{r_2}2\pi r dr U^2+\tfrac{1}{2}mdV^2$$

$$=\;2\pi^2Rd.\int_{r_1}^{r_2}\left(\frac{A_0}{\sqrt{2}.ar}+\frac{Aar}{2}\right)^2rdr+\tfrac{1}{2}mdV^2$$

$$=2\pi^2Rd\left\{\tfrac{1}{2}\frac{A_0^2}{a^2}\log\frac{r_2}{r_1}+\frac{A_0A}{2\sqrt{2}}(r_2^2-r_1^2)+\frac{A^2a^2}{16}(r_2^4-r_1^4)\right\}+\tfrac{1}{2}mdV^2$$

Now

$$\mathrm{A}_0=-\frac{\mu_1 a}{\pi\sqrt{2}}+\frac{\mu' a}{\pi\sqrt{2}}\frac{r_1^2}{r_2^2-r_1^2}$$

$$\mathrm{A}a=-\frac{\mu'}{4\pi a^2(k_2^2-k_1^2)}=-\frac{\mu'}{\pi(r_2^2-r_1^2)}$$

Hence

$$\mathrm{E}_1=\tfrac{1}{4}\mathrm{R}d\left\{\left(\mu_1-\frac{x}{1-x}\mu'\right)^2\log\frac{1}{x}+2\mu'\left(\mu_1-\frac{x}{1-x}\mu'\right)+\tfrac{1}{2}\mu'^2\frac{1+x}{1-x}\right\}+\tfrac{1}{2}md\mathrm{V}^2 \quad . \quad . \quad (49)$$

Here the terms in μ' are small compared with the others, even in the case where x is nearly unity. Hence the principal part of E_1 is

$$\mathrm{E}_1=\tfrac{1}{4}\mathrm{R}d\mu_1^2\log\frac{1}{x}=\tfrac{1}{2}d\mu_1^2 a(\mathrm{L}_1-\mathrm{L}_2)$$

To the same order E_2 has been found in the paper "On the Steady Motion of a Hollow Vortex" [I., § 12]. Using the result there obtained

$$\mathrm{E}_2=2\pi^2 r_2[\mathrm{U}'_2\psi_0+\tfrac{1}{2}\mathrm{U}'_2\mathrm{V}(\mathrm{R}^2-\lambda a^2+\tfrac{1}{2}r^2)+\tfrac{1}{2}\mathrm{R}r\mathrm{V}^2]d'$$

where

$$\mathrm{U}_2'=\frac{\mu_2}{4\pi a k_2}=\frac{\mu_2}{2\pi r_2}$$

$$\psi_0=\frac{\mu_2 a}{2\pi}(\mathrm{L}_2-2)$$

Further, since $\lambda=1-4(\mathrm{L}_2-2)k_2^2$, $r_2=2ak_2$, $\mathrm{R}=a(1+2k_2^2)$*

$$\mathrm{R}^2-\lambda a^2+\tfrac{1}{2}r^2=4(\mathrm{L}_2-\tfrac{1}{2})a^2k_2^2=(\mathrm{L}_2-\tfrac{1}{2})r_2^2$$

Hence

$$\mathrm{E}_2=2\pi^2 r_2\left[\frac{\mu_2^2 a}{4\pi^2 r_2}(\mathrm{L}_2-2)+\frac{\mu_2\mathrm{V}}{4\pi}r_2(\mathrm{L}_2-\tfrac{1}{2})+\tfrac{1}{2}\mathrm{R}r_2\mathrm{V}^2\right]d'$$

$$=\mathrm{M}'\left[\frac{\mu_2^2(1-2k_2^2)}{4\pi^2 r_2^2}(\mathrm{L}_2-2)+\frac{\mu_2\mathrm{V}}{4\pi\mathrm{R}}(\mathrm{L}_2-\tfrac{1}{2})+\tfrac{1}{2}\mathrm{V}^2\right] \quad . \quad . \quad . \quad . \quad (50)$$

where M′ is the mass of the outer fluid displaced by the ring. The most important term in this is

$$\mathrm{E}_2=\mathrm{M}'\frac{\mu_2^2}{4\pi^2 r_2^2}(\mathrm{L}_2-2)=\tfrac{1}{2}d'\mu_2^2 a(\mathrm{L}_2-2)$$

E_1 is in general of order k^2 with respect to this, unless there is an added inner circulation μ_1.

When there is no added internal circulation, the energy of the ring itself is small compared with the whole. When, however, there is such a circulation, and when the hollow is small, the internal energy is large, and may be larger than the external. As, however, the aperture increases—or as we have already seen the hollow increases—the internal energy will rapidly diminish. The effect therefore of increasing the whole energy is to actually decrease that of the core.

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* See erratum at end.

The external energy will, as in the ordinary theory, increase with the aperture.

Section III.—*Small Vibrations.*

13. The vibrations here considered are (1) when the core is fluted, and (2) when it pulsates. The normal modes can be represented by principal displacements of the form $\alpha_n \cos nv$, together with a series of terms whose amplitudes are (*with respect to k*) infinitesimally smaller than α_n. The time of vibration will be a function of a, k_1, k_2. and may therefore be supposed expressed as a series in terms of k_1, k_2. It will be the aim in what follows to obtain the first, or principal term, in this expansion, in other words,—in functions of k_1, k_2—, k_1 k_2 will be regarded as small quantities of the first order. To this order it will be found that we need only consider the principal term in the normal mode concerned.

Let the form of the surface at any time be given by

$$\left.\begin{aligned} k&=k_1(1+\alpha_n \cos nv+\gamma_n \sin nv) \\ k&=k_2(1+\beta_n \cos nv+\delta_n \sin nv) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

where α, β, γ, δ are functions of the time.

The motion is then determined by the stream functions already obtained, with an added stream function, or velocity potential, to give the small motions.

In general, the stream function for the vibrations will be many valued. It has however been shown (I., § 13) that in the case of fluted vibrations this portion is of order k^{n+1}, and may be neglected. This cannot, however, be done in the case of pulsations, and for that the velocity potential must be employed.

Let χ_1, χ_2 be the additional stream functions.

Then

$$\chi_1=\frac{1}{\sqrt{(\mathrm{C}-c)}}\{(\mathrm{A}_n\mathrm{R}_n+\mathrm{B}_n\mathrm{T}_n)\cos nv+(\mathrm{C}\mathrm{R}_n+\mathrm{D}_n\mathrm{T}_n)\sin nv\}+\epsilon$$

$$\chi_2=\frac{1}{\sqrt{(\mathrm{C}-c)}}\{\mathrm{A}'_n\mathrm{R}_n\cos nv+\mathrm{C}'_n\mathrm{R}_n\sin nv\}+\epsilon'$$

where ϵ, ϵ' are uniform along the boundary.

The expression for the pressure at any point of the core has been already found (11) viz.:

$$\frac{p}{d}=\mathrm{C}+f(t)-\dot{\phi}-\tfrac{1}{2}(\mathrm{vel})^2+\mathrm{A}\psi.$$

Since it will be necessary to obtain the value of $\dot{\phi}$, it will be necessary to obtain the value of ψ at any time; that is when the boundaries are given by equation (51). This will consist of two portions, one determined by the normal motion at the boundary, and the other (ϵ) by the fact that the circulation remains unaltered.

14. To find the function χ it is first necessary to know the velocity normal to the boundary at any time. Now since the boundary is not the circle, it is in steady motion, the function ψ_0 will itself produce a normal motion. This must first be found. Let θ be the angle which the boundary at any point makes with the circle $u=u$, then

$$\tan\theta=\frac{dn}{dn'}=\frac{du}{dv}=-\frac{1}{k}\frac{dk}{dv}$$

or since

$$k=k_1(1+\alpha\cos nv+\gamma\sin nv)$$

$$\tan\theta=n(\alpha\sin nv-\gamma\cos nv)\quad\ldots\ldots\ldots\quad(52)$$

Now the normal velocity outwards is

$$=-\frac{1}{\rho}\frac{\delta\psi}{\delta v}\frac{dv}{dn}\cos\theta-\frac{1}{\rho}\frac{\delta\psi}{\delta u}\frac{du}{dn}\sin\theta$$

where $\psi=\psi_0+\chi$, ψ_0 being the stream function already found. Also, remembering that θ is small, the normal velocity is

$$=-\frac{dn}{dt}\cos\theta-\mathrm{U}\sin\theta$$

$$=2ak_1(\dot{\alpha}\cos nv+\dot{\gamma}\sin nv)-n\mathrm{U}(\alpha\sin nv-\gamma\cos nv)$$

Therefore, along the boundary

$$-\frac{1}{\rho}\frac{\delta\chi}{dv}\frac{dv}{dn}=(2ak_1\dot{\alpha}+n\mathrm{U}\gamma)\cos nv+(2ak_1\dot{\gamma}-n\mathrm{U}\alpha)\sin nv$$

Then the two sets of conditions give (letters with one dash referring to the inner surface, and with two dashes to the outer surface)

$$\left.\begin{aligned}\mathrm{AR}'_n+\mathrm{BT}'_n&=-a^2\mathrm{U}_1\sqrt{(2k_1)}\alpha+\frac{a^3}{n}(2k_1)^{\frac{3}{2}}\dot{\gamma}\\ \mathrm{AR}''_n+\mathrm{BT}''_n&=-a^2\mathrm{U}_2\sqrt{(2k_2)}\beta+\frac{a^3}{n}(2k_2)^{\frac{3}{2}}\dot{\delta}\\ \mathrm{CR}'_n+\mathrm{DT}'_n&=-a^2\mathrm{U}_1\sqrt{(2k_1)}\gamma-\frac{a^3}{n}(2k_1)^{\frac{3}{2}}\dot{\alpha}\\ \mathrm{CR}''_n+\mathrm{DT}''_n&=-a^2\mathrm{U}_2\sqrt{(2k_2)}\delta-\frac{a^3}{n}(2k_2)^{\frac{3}{2}}\dot{\beta}\end{aligned}\right\}\quad\ldots\ldots\quad(53)$$

whilst for the motion outside the core

$$\left.\begin{aligned} \mathsf{A}\mathrm{R}''_n &= -a^2\mathrm{U}'_2\sqrt{(2k_2)}\beta+\frac{a^3}{n}(2k_2)^{\frac{3}{2}}\dot{\delta} \\ \mathsf{C}\mathrm{R}''_n &= -a^2\mathrm{U}'_2\sqrt{(2k_2)}\delta-\frac{a^3}{n}(2k_2)^{\frac{3}{2}}\dot{\beta} \end{aligned}\right\} \quad \ldots \ldots \quad (54)$$

Denote the right hand members of these by ξ, η, ξ', η', η_1, η'_1, then solving

$$\mathrm{A}=\frac{\xi\mathrm{T}''_n-\eta\mathrm{T}'_n}{\mathrm{R}'_n\mathrm{T}''_n-\mathrm{R}''_n\mathrm{T}'_n}$$

$$\mathrm{B}=\frac{-\xi\mathrm{R}''_n+\eta\mathrm{R}'_n}{\mathrm{R}'_n\mathrm{T}''_n-\mathrm{R}''_n\mathrm{T}'_n}$$

with similar expressions for C, D, in ξ', η'.

Suppose now

$$\epsilon=\frac{1}{\sqrt{\mathrm{C}-c}}\Sigma\{(\mathrm{A}_n\mathrm{R}_n+\mathrm{B}_n\mathrm{T}_n)\cos nv+(\mathrm{C}_n\mathrm{R}_n+\mathrm{D}_n\mathrm{T}_n)\sin nv\}$$

It has to be determined from the fact that ϵ is uniform along the boundary, *i.e.*, when $k=k_1(1+\alpha_n\cos nv+\gamma_n\sin nv)$, &c., and that the whole circulation remains unchanged.

If U denote the velocity along u, at a point of the boundary; V the velocity along v as determined by ψ alone, and θ the inclination of the boundary to u (and therefore a small angle); then the circulation is

$$\mu=\frac{\pi\sqrt{2}}{a}(\mathrm{A}+\Sigma\mathrm{A}_n)+\int(\mathrm{U}\cos\theta+\mathrm{V}\sin\theta)ds$$

Now

$$\mathrm{U}\cos\theta+\mathrm{V}\sin\theta=\left(\mathrm{U}_0+\frac{d\mathrm{U}_0}{dk}\delta k\right)\left(1-\frac{\theta^2}{2}\right)+\frac{d\mathrm{V}}{dk}\delta k.\theta.$$

$$=\mathrm{U}_0+\frac{d\mathrm{U}_0}{dk}\delta k+\frac{d\mathrm{V}}{dk}\delta k\theta.$$

Now θ is of order δk. Also $\frac{d\mathrm{V}}{dk}$ is of higher order than we require; for both reasons therefore it may be neglected, and

$$\int(\mathrm{U}\cos\theta+\mathrm{V}\sin\theta)ds=\mu+\frac{d\mathrm{U}_0}{dk}\int_0^{2\pi}k(\alpha\cos nv+\gamma\sin nv)\frac{adv}{\mathrm{C}-c}$$

since

$$\frac{dn}{dv}=\frac{a}{\mathrm{C}-c}$$

Also

$$\begin{aligned}&\frac{d\mathrm{U}_0}{dk}\,ak\int_0^{2\pi}\frac{\alpha\cos nv+\gamma\sin nv}{\mathrm{C}-c}\,dv\\
&=ak\frac{d\mathrm{U}_0}{dk}\int_0^{2\pi}\alpha\cos^2 nv\frac{2e^{-nu}}{\mathrm{S}}\,dv\\
&=\frac{2\pi ak\alpha e^{-nu}}{\mathrm{S}}\cdot\frac{d\mathrm{U}_0}{dk}\\
&=4\pi a\alpha k^{n+2}\frac{d\mathrm{U}_0}{dk}\end{aligned}$$

and can therefore be neglected except in the case of pulsations.

Hence $\mathrm{A}+\Sigma\mathrm{A}_n\mathrm{R}_n=0$ to our order of approximations. The terms in ϵ can therefore be neglected except in pulsations, *i.e.*, $n=0$.

15. The next step is to determine $\dot{\phi}$. Now ϕ is the flow along any line up to the point in question. Choose this line to be the two portions

(1) straight line $v=\pi$ from $u=u_1$ to $u=u$
(2) circle $u=u$ from $v=v$ to $v=\pi$

The part of ϕ depending on ψ_0 depends on the path of integration, but is constant, and therefore will disappear in $\dot{\phi}$. Consequently the part depending on χ is the only portion needed. Hence

$$\begin{aligned}\phi&=\int_{u_1}^{u}\left[\frac{1}{\rho}\frac{d\chi}{dv}\frac{dv}{dn'}\frac{dn}{du}\right]_{v=\pi}du+\int_{v}^{\pi}\left[\frac{1}{\rho}\frac{d\chi}{du}\frac{du}{dn}\frac{dn'}{dv}\right]_{u=u}dv\\
&=\int_{u_1}^{u}\left[\frac{1}{\rho}\frac{d\chi}{dv}\right]_{v=\pi}du+\int_{v}^{\pi}\left[\frac{1}{\rho}\frac{d\chi}{du}\right]_{u=u}dv\\
&=\phi_1+\phi_2\text{ (say)}\end{aligned}$$

Now

$$\chi=\frac{1}{\sqrt{(\mathrm{C}-c)}}(\mathrm{L}\cos nv+\mathrm{M}\sin nv)\text{ (say)}$$

Hence

$$\begin{aligned}\phi_1&=(-)^n n\int_{u_1}^{u}\frac{\sqrt{(\mathrm{C}+1)}}{a\mathrm{S}}\mathrm{M}\,du\\
&=(-)^{n+1}\frac{n}{a}\int_{k'}^{k}\sqrt{\left(\frac{2}{k}\right)}\mathrm{M}\,dk\\
&=(-)^{n+1}\frac{n}{a}\int_{k'}^{k}\sqrt{\left(\frac{2}{k}\right)}(\mathrm{CR}_n+\mathrm{DT}_n)\,dk\end{aligned}$$

Whence, as the lowest terms only are required, and as $\mathrm{R}_n\propto k^{-\frac{2n+1}{2}}$ and $\mathrm{T}_n\propto k^{\frac{2n-1}{2}}$

$$\phi_1=(-)^n\left[\frac{\sqrt{(2k)}}{a}(\mathrm{CR}_n-\mathrm{DT}_n)\right]_{u_1}^{u}$$

Again,

$$\frac{d\chi}{du}=\surd(2k)\left\{\left(\frac{d\mathrm{L}}{du}-\tfrac{1}{2}\mathrm{L}\right)\cos nv+\left(\frac{d\mathrm{M}}{du}-\tfrac{1}{2}\mathrm{M}\right)\sin nv\right\}$$

Therefore

$$\phi_2=\frac{\surd(2k)}{a}\int_v^\pi\left\{\left(\frac{d\mathrm{L}}{du}-\tfrac{1}{2}\mathrm{L}\right)\cos nv+\left(\frac{d\mathrm{M}}{du}-\tfrac{1}{2}\mathrm{M}\right)\sin nv\right\}dv$$

$$=\frac{\surd(2k)}{na}\left\{-\left(\frac{d\mathrm{L}}{du}-\tfrac{1}{2}\mathrm{L}\right)\sin nv+\left(\frac{d\mathrm{M}}{du}-\tfrac{1}{2}\mathrm{M}\right)(\cos nv-(-1)^n)\right\}$$

Now

$$\mathrm{L}=\mathrm{AR}_n+\mathrm{BT}_n$$

Therefore

$$\frac{d\mathrm{L}}{du}-\tfrac{1}{2}\mathrm{L}=\mathrm{A}\left(\frac{d\mathrm{R}_n}{du}-\tfrac{1}{2}\mathrm{R}_n\right)+\mathrm{B}\left(\frac{d\mathrm{T}_n}{du}-\tfrac{1}{2}\mathrm{T}_n\right)$$

$$=\mathrm{A}\{(n^2-\tfrac{1}{4})\mathrm{SP}_n-\tfrac{1}{2}\mathrm{R}_n\}+\mathrm{B}\{-(n^2-\tfrac{1}{4})\mathrm{SQ}_n-\tfrac{1}{2}\mathrm{T}_n\}$$

$$=\mathrm{AR}_n\left\{\frac{2n\mathrm{SP}_n}{\mathrm{P}_{n+1}-\mathrm{P}_{n-1}}-\tfrac{1}{2}\right\}-\mathrm{BT}_n\left\{\frac{2n\mathrm{SQ}_n}{\mathrm{Q}_{n-1}-\mathrm{Q}_{n+1}}+\tfrac{1}{2}\right\}$$

Now $\mathrm{P}_n/\mathrm{P}_{n+1}$ and $\mathrm{Q}_n/\mathrm{Q}_{n-1}$ are both of order k.
Also,

$$(2n+1)\mathrm{P}_{n+1}-4n\mathrm{CP}_n+(2n-1)\mathrm{P}_{n-1}=0$$

Therefore

$$2n\frac{\mathrm{P}_n}{\mathrm{P}_{n+1}}=\frac{2n+1}{2\mathrm{C}}+\frac{2n-1}{2\mathrm{C}}\frac{\mathrm{P}_{n-1}}{\mathrm{P}_{n+1}}$$

or to the lowest order

$$2n\frac{\mathrm{P}_n}{\mathrm{P}_{n+1}}=(2n+1)k$$

So

$$2n\frac{\mathrm{Q}_n}{\mathrm{Q}_{n-1}}=(2n-1)k$$

and therefore to the same order

$$\frac{d\mathrm{L}}{du}-\tfrac{1}{2}\mathrm{L}=n(\mathrm{AR}_n-\mathrm{BT}_n)\,.\quad\quad(55)$$

and similarly

$$\frac{d\mathrm{M}}{du}-\tfrac{1}{2}\mathrm{M}=n(\mathrm{CR}_n-\mathrm{DT}_n)$$

Hence

$$\phi_1+\phi_2=\text{ const. }+\frac{\surd(2k)}{a}\{-(\mathrm{AR}_n-\mathrm{BT}_n)\sin nv+(\mathrm{CR}_n-\mathrm{DT}_n)\cos nv\}\quad.\quad(56)$$

The part of ϕ_1 containing u disappearing with a corresponding term in ϕ_2, as it ought consistently with the fact that this part of ϕ is independent of the path of integration.

The values of $\dot{\phi}$ are needed along the inner and outer surfaces. Hence substituting the values of A, B, C, D already found, along the inner surface

$$\dot{\phi}=\frac{\sqrt{(2k_1)}}{a}\left\{-\frac{\dot{\xi}(\mathrm{R}'_n\mathrm{T}''_n+\mathrm{R}''_n\mathrm{T}'_n)-2\dot{\eta}\mathrm{R}'_n\mathrm{T}'_n}{\mathrm{R}'_n\mathrm{T}''_n-\mathrm{R}''_n\mathrm{T}'_n}\sin nv+\frac{\dot{\xi}'(\mathrm{R}'_n\mathrm{T}''_n+\mathrm{R}''_n\mathrm{T}'_n)-2\dot{\eta}'\mathrm{R}'_n\mathrm{T}'_n}{\mathrm{R}'_n\mathrm{T}''_n-\mathrm{R}''_n\mathrm{T}'_n}\cos nv\right\}$$

Now the lowest terms in R_n, T_n are proportional respectively to $k^{-\frac{2n+1}{2}}$, $k^{\frac{2n-1}{2}}$. Hence to the same order

$$\dot{\phi}=\frac{\sqrt{2}}{a}\left\{-\left(\frac{k_2^{2n}+k_1^{2n}}{k_2^{2n}-k_1^{2n}}\dot{\xi}\sqrt{k_1}-\frac{2(k_1k_2)^n}{k_2^{2n}-k_1^{2n}}\dot{\eta}\sqrt{k_2}\right)\sin nv+(\ldots)\cos nv\right\}$$

We shall put for the future

$$\frac{k_2^{2n}+k_1^{2n}}{k_2^{2n}-k_1^{2n}}=p,\quad \frac{2(k_1k_2)^n}{k_2^{2n}-k_1^{2n}}=q$$

where it is to be noticed that

$$p^2-q^2=1$$

With this notation

$$\dot{\phi}=\frac{\sqrt{2}}{a}\{-(p\dot{\xi}\sqrt{k_1}-q\dot{\eta}\sqrt{k_2})\sin nv+(p\dot{\xi}'\sqrt{k_1}-q\dot{\eta}'\sqrt{k_2})\cos nv\}\quad . \quad . \quad (57)$$

So along the outside boundary

$$\dot{\phi}=\frac{\sqrt{2}}{a}\{-(q\dot{\xi}\sqrt{k_1}-p\dot{\eta}\sqrt{k_2})\sin nv+(q\dot{\xi}'\sqrt{k_1}-p\dot{\eta}'\sqrt{k_2})\cos nv\}\quad . \quad . \quad (58)$$

whilst for the outside boundary in the outside fluid

$$\dot{\phi}=\frac{\sqrt{(2k_2)}}{a}(-\dot{\eta}_1\sin nv+\dot{\eta}_1'\cos nv)\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

16. *The pressure conditions.*—The expression for the pressure at any point of the core has been already found (11), viz. :—

$$\frac{p}{d}=\text{const}+f(t)-\dot{\phi}-\tfrac{1}{2}v^2+\mathrm{A}\psi$$

At a point near a surface this is

$$\begin{aligned}\frac{p}{d}&=\text{const}+f(t)-\dot{\phi}-\tfrac{1}{2}\left(\mathrm{U}+\frac{\delta\mathrm{U}}{\delta k}\delta k+\frac{\mathrm{S}}{\rho^2}\frac{\delta\chi}{\delta u}\right)^2-\tfrac{1}{2}\left(\frac{\mathrm{S}}{\rho^2}\frac{\delta\psi}{\delta v}+\frac{\mathrm{S}}{\rho^2}\frac{\delta\chi}{\delta v}\right)^2+\mathrm{A}\left(\psi_0+\frac{\delta\psi}{\delta k}\delta k+\chi\right)\\&=\text{const}+f(t)-\dot{\phi}-\tfrac{1}{2}\mathrm{U}^2-\mathrm{U}\left(\frac{\delta\mathrm{U}}{\delta k}\delta k+\frac{1}{2a^2k}\frac{\delta\chi}{\delta u}\right)+\mathrm{A}(\psi_0-2a^2\mathrm{U}\delta k+\chi)\end{aligned}$$

U being the tangential velocity along the original surface.

Now

$$\mathrm{U}=-\frac{1}{2a^2}\frac{\delta\psi}{\delta k}$$

Hence outside

$$a^2\mathrm{U}=-\frac{\mathrm{A}_0'}{2\sqrt{2k}}$$

$$\frac{d\mathrm{U}_2'}{dk}=-\frac{\mathrm{U}_2'}{k} \quad \text{. . . . .} \quad (60)$$

and inside

$$a^2\mathrm{U}=-\frac{\mathrm{A}_0}{2\sqrt{2k}}-\mathrm{A}a^4k$$

$$\frac{d\mathrm{U}}{dk}=-\frac{\mathrm{U}}{k}-2\mathrm{A}a^2 \quad \text{. . . . .} \quad (61)$$

also

$$\frac{\delta\chi}{du}=\sqrt{(2k)}\left\{\left(\frac{d\mathrm{L}}{du}-\tfrac{1}{2}\mathrm{L}\right)\cos nv+\left(\frac{d\mathrm{M}}{du}-\tfrac{1}{2}\mathrm{M}\right)\sin nv\right\}$$

whence the part of the pressure depending on v is

$$\frac{p}{d}=-\dot{\phi}-\mathrm{U}\left\{-\left(\frac{\mathrm{U}}{k}+2\mathrm{A}a^2\right)\delta k+\frac{n}{a^2\sqrt{(2k)}}(\mathrm{AR}_n-\mathrm{BT}_n)\cos nv\right.$$
$$\left.+\frac{n}{a^2\sqrt{(2k)}}(\mathrm{CR}_n-\mathrm{DT}_n)\sin nv\right\}+\mathrm{A}(\chi-2a^2\mathrm{U}\delta k)$$

Along the inside surface this vanishes. Therefore

$$\frac{1}{a}\{p\dot{\xi}\sqrt{(2k_1)}-q\dot{\eta}\sqrt{(2k_2)}\}+\mathrm{U}_1(\mathrm{U}_1+2\mathrm{A}a^2k_1)\gamma-\frac{n\mathrm{U}_1}{2a^2k_1}\{p\xi'\sqrt{(2k_1)}-q\eta'\sqrt{(2k_2)}\}$$
$$-\mathrm{A}\left\{\frac{4a^3k_1^{\,2}}{n}\dot{\alpha}+4a^2k_1\mathrm{U}_1\gamma\right\}=0$$

$$-\frac{1}{a}\{p\dot{\xi}'\sqrt{(2k_1)}-q\dot{\eta}'\sqrt{(2k_2)}\}+\mathrm{U}_1(\mathrm{U}_1+2\mathrm{A}a^2k_1)\alpha-\frac{n\mathrm{U}_1}{2a^2k}\{p\xi\sqrt{(2k_1)}-q\eta\sqrt{(2k_2)}\}$$
$$+\mathrm{A}\left\{\frac{4a^3k_1^{\,2}}{n}\dot{\gamma}-4a^2k_1\mathrm{U}_1\alpha\right\}=0$$

Substituting for ξ, η, &c., these become

$$-\frac{4\mathrm{A}a^3k_1^{\,2}}{n}\dot{\alpha}+\frac{4a^2k_1^{\,2}}{n}p\ddot{\gamma}+\{\mathrm{U}_1(\mathrm{U}_1-2\mathrm{A}a^2k_1)+n\mathrm{U}_1^{\,2}p\}\gamma$$
$$+\left(2ak_2q\mathrm{U}_2-2a\frac{k_2^{\,2}}{k_1}q\mathrm{U}_1\right)\dot{\beta}-\frac{4a^2k_2^{\,2}}{n}q\ddot{\delta}-n\mathrm{U}_1\mathrm{U}_2\frac{k_2}{k_1}q\delta=0$$

with a corresponding expression from the second equation. In these equations alter the meaning of the α, so as to write α for $k_1^2\alpha$, γ for $k_1^2\gamma$, β for $k_2^2\beta$, δ for $k_2^2\delta$. Further, write u, v, w for $\mathrm{U}_1/(2ak_1)$, $\mathrm{U}_2/(2ak_2)$, $\mathrm{U}'_2/(2ak_2)$ respectively. Then these equations become

$$\left.\begin{array}{r} -\mathrm{A}a\dot{\alpha}+p\ddot{\gamma}+\{n(np+1)u-n\mathrm{A}a\}u\gamma+nq(v-u)\dot{\beta}-q\ddot{\delta}-n^2uvq\delta=0 \\ p\ddot{\alpha}+\{n(np+1)u-n\mathrm{A}a\}u\alpha+\mathrm{A}a\dot{\gamma}-q\ddot{\beta}-n^2quv\beta-nq(v-u)\dot{\delta}=0 \end{array}\right\} . \qquad (62)$$

Along the outside surface, we find the corresponding expressions by interchanging k_1, k_2; α, β; γ, δ; u, v, *i.e.*, put $-p$ for p and $-q$ for q.

Then

$$\frac{\mathrm{P}}{d}=\frac{4a^2}{n}[\{-p\ddot{\beta}+(\overline{-np+1}|v-\mathrm{A}a)nv\beta+\mathrm{A}a\dot{\delta}+q\ddot{\alpha}+n^2uvq\alpha-nq(v-u)\dot{\gamma}\}\cos nv$$

$$+\{-\mathrm{A}a\dot{\beta}-p\ddot{\delta}+(\overline{-np+1}|v-\mathrm{A}a)nv\delta+nq(v-u)\dot{\alpha}+q\ddot{\gamma}+n^2uvq\gamma\}\sin nv]$$

Again

$$\frac{\mathrm{P}}{d'}=-\frac{\sqrt{(2k_2)}}{a}(\dot{\eta}'_1\cos nv-\dot{\eta}_1\sin nv)-\mathrm{U}'_2\left(\frac{d\mathrm{U}'_2}{dk}\delta k+\frac{1}{2a^2k}\frac{\delta\chi'}{\delta u}\right)+\text{const}$$

$$=-\frac{\sqrt{(2k_2)}}{a}(\dot{\eta}'_1\cos nv-\dot{\eta}_1\sin nv)$$

$$-\mathrm{U}'_2\left\{-\mathrm{U}'_2(\beta\cos nv+\delta\sin nv)+\frac{n}{2a^2k_2}\sqrt{(2k_2)}(\eta_1\cos nv+\eta_1\sin nv)\right\}$$

Substituting for the η from (54), and altering the meaning of α, β, γ, δ as before

$$\frac{\mathrm{P}}{d'}=\frac{4a^2}{n}[\{\ddot{\beta}+n(n+1)w^2\beta\}\cos nv+\{\dot{\delta}+n(n+1)w^2\delta\}\sin nv]$$

Hence, since the pressure is the same on both sides the boundary

$$\left.\begin{array}{l} q\ddot{\alpha}+n^2quv\alpha-nq(v-u)\dot{\gamma}-p\ddot{\beta}+n\{(-np+1)v-\mathrm{A}a\}v\beta+\mathrm{A}a\dot{\delta} \\ \qquad =\dfrac{d'}{d}\{\ddot{\beta}+n(n+1)w^2\beta\} \\ nq(v-u)\dot{\alpha}+q\ddot{\gamma}+n^2quv\gamma-\mathrm{A}a\dot{\beta}-p\ddot{\delta}+n\{(-np+1)v-\mathrm{A}a\}v\delta \\ \qquad =\dfrac{d'}{d}\{\ddot{\delta}+n(n+1)w^2\delta\} \end{array}\right\}$$

When the waves are travelling in the positive direction round the ring, δk will be of the form

$$\delta k_1=\mathrm{L}\sin(nv+\lambda t)$$

$$\delta k_2=\mathrm{M}\sin(nv+\lambda t)$$

or

$$\alpha=\mathrm{L}\sin\lambda t \quad , \quad \gamma=\mathrm{L}\cos\lambda t$$

$$\beta=\mathrm{M}\sin\lambda t \quad , \quad \delta=\mathrm{M}\cos\lambda t$$

Substituting these values, and writing ρ for d'/d, the four equations of condition reduce to the two

$$\left.\begin{array}{l} \mathrm{L}\{p\lambda^2+\mathrm{A}a\lambda-n(np+1)u^2+n\mathrm{A}au\}-\mathrm{M}q(\lambda-nu)(\lambda+nv)=0 \\ q\mathrm{L}(\lambda+nu)(\lambda-nv)-\mathrm{M}\{(p+\rho)\lambda^2-\mathrm{A}a\lambda-n(np-1)v^2-n\mathrm{A}av-n(n+1)w^2\rho\}=0 \end{array}\right\} \quad (62a.)$$

Whence the equation to determine λ is

$$\left.\begin{vmatrix} p\lambda^2+\mathrm{A}a\lambda-n(np+1)u^2+n\mathrm{A}au\ , & q(\lambda-nu)(\lambda+nv) \\ q(\lambda+nu)(\lambda-nv)\ , & (p+\rho)\lambda^2-\mathrm{A}a\lambda-n(np-1)v^2-n\mathrm{A}av-n(n+1)w^2\rho \end{vmatrix}=0\right\} \quad (63)$$

17. Case I.—No rotational core. Here the periods are given by

$$\lambda^2-n(n+1)w^2=0.$$

The time of vibration is therefore

$$\left.\begin{array}{rl} & \dfrac{2\pi}{w\sqrt{n(n+1)}}=\dfrac{(4\pi ak_2)^2}{\mu_2\sqrt{n(n+1)}} \\ \text{or by (39)} & \\ & =\dfrac{\mu_2 d'}{2\Pi\sqrt{n(n+1)}} \end{array}\right\} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (64)$$

and is therefore the same whatever the size of the ring. Moreover, when n is large, the time varies inversely as n. The ring is always stable for vibrations of this mode. The expression for the time does not agree with that found in the former paper [I. § 13]. The reason of this is stated in a note at the end of the present paper.

Case II. *Continuous case.*—Here $\mu_1=0$, $k_1=0$, $\alpha=\gamma=0$, $p=1$, $q=0$, and the equations of motion reduce to two

$$\left.\begin{array}{l} -(1+\rho)\ddot{\beta}-\{n(n-1)v^2+n(n+1)w^2\rho+n\mathrm{A}av\}\beta+\mathrm{A}a\dot{\delta}=0 \\ -\mathrm{A}a\dot{\beta}-(1+\rho)\ddot{\delta}-\{n(n-1)v^2+n(n+1)w^2\rho+n\mathrm{A}av\}\delta=0 \end{array}\right\}$$

Here putting

$$\delta k=\mathrm{L}\sin(nv+\lambda t)$$

$$(1+\rho)\lambda^2-\mathrm{A}a\lambda-\{n(n-1)v^2+n(n+1)w^2\rho+n\mathrm{A}av\}=0.$$

Since the core is continuous

$$\mu'=-4\pi\mathrm{A}a^3k_2^{\ 2}$$

therefore

$$\mathrm{A}a=-\frac{\mu'}{4\pi a^2k_2^{\ 2}}=-2v$$

whence

$$\lambda^2+\frac{2v}{1+\rho}\lambda=\frac{n\{(n-3)v^2+(n+1)w^2\rho\}}{1+\rho}$$

and

$$\lambda=\frac{-v\pm\sqrt{\{v^2+n(1+\rho)((n-3)v^2+(n+1)w^2\rho)\}}}{1+\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

In order that the motion may be stable the expression under the root must always be positive, whatever value n has, $= > 1$. The least value is when $n=1$. Hence the condition of stability is

$$2w^2\rho(1+\rho)>(1+2\rho)v^2$$

or

$$\mu_2{}^2>\mu'^2\frac{1+2\rho}{2(1+\rho)\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

In the ordinary theory where $\rho=1$ and $\mu_2=\mu'$ the ring is therefore always stable. When there is no additional circulation the condition becomes $\rho>1/\sqrt{2}$ or $d<d'\sqrt{2}$. The ordinary simple ring will therefore be stable even when the density of its core is as large as $\sqrt{2}$ times the surrounding fluid. In any case, whatever the density of the core may be, the ring will be stable provided it has an additional outside circulation given by (66). In general the two values of λ will be of opposite sign, and will correspond to waves travelling with and against the cyclic motion. The positive root gives that in the same direction and the negative in the opposite. In the simplest cases of equal density and no added circulation the roots are $-nv$ and $(n-1)v$.

18. Case III.—*No internal added circulation*, with a hollow. Here $\mu_1=0$, $u=0$, and the equation for λ gives one root zero, and the cubic

$$(1+p\rho)\lambda^3+\mathrm{A}a\rho\lambda^2-\{n(n-p)v^2+np(n+1)w^2\rho+n\mathrm{A}apv+\mathrm{A}^2a^2\}\lambda$$
$$-n\mathrm{A}a\{(np-1)v^2+(n+1)w^2\rho+\mathrm{A}av\}=0$$

in which since

$$\mu'=-4\pi\mathrm{A}a^3(k_2{}^2-k_1{}^2)$$

$$\mathrm{A}a=-\frac{2v}{1-x}$$

Denote the cubic by

$$\lambda^3+b\lambda^2+c\lambda+d=0$$

We shall first investigate the signs and finiteness of the different coefficients, and then pass on to the question of the reality of the roots.

$$b=\frac{\mathrm{A}a\rho}{1+p\rho}=-\frac{2v\rho}{1-x}\times\frac{1-x^n}{1-x^n+(1+x^n)\rho}$$

Since

$$p=\frac{k_2^{2n}+k_1^{2n}}{k_2^{2n}-k_1^{2n}}=\frac{1+x^n}{1-x^n}$$

Further denote $1+x+\ldots+x^{n-1}$ by X_n and $(1-x^n)+(1+x^n)\rho$ by f_n. It is to be noticed that since x lies between 0 and 1, these quantities are both positive and finite.

With this notation

$$b=-\frac{2\rho X_n}{f_n}v$$

and is always negative and finite.

The second coefficient is

$$c=-\frac{1}{1+p\rho}\{np(n+1)w^2\rho+n(n-p)v^2+nAapv+A^2a^2\}$$

After some algebraic reduction this may be put in the form

$$c=-\frac{1}{f_n}\left[(1+x^n)n(n+1)w^2\rho+(1-x)v^2\Sigma_0^{n-1}\left\{\left(n-\frac{2r+3}{2}\right)^2+r^2+3r+\tfrac{7}{4}\right\}x^r\right]$$

Here c is always finite and negative.

The third coefficient is

$$\begin{aligned}d&=-\frac{nAa}{1+p\rho}\{(n+1)w^2\rho+(np-1)v^2+Aav\}\\&=\frac{2nX_n}{f_n}\left[(n+1)(w^2\rho-v^2)+\left\{\frac{2n}{1-x^n}-\frac{2}{1-x}\right\}v^2\right]v\end{aligned}$$

which again, after some algebraic reduction, becomes

$$d=\frac{2nX_n}{f_n}\left\{(n+1)(w^2\rho-v^2)+\frac{2v^2}{X_n}\Sigma_0^{n-2}(n-r-1)x^r\right\}v$$

This is always finite, and if $w^2\rho>v^2$ it is also always positive. But if $w^2\rho<v^2$ it becomes negative for values of n less than a particular value, depending on the value of $w^2\rho/v^2$.

The conditions that

$$\lambda^3+b\lambda^2+c\lambda+d=0$$

shall have real roots are

i. $b^2>3c$

ii. $b^2c^2-4c^3>(4b^3-18bc+27d)d$

We have already seen that c is always negative. Hence the first condition is satisfied under all circumstances.

The second condition is

$$-4c^3+b^2c^2+18bcd-4b^3d>27d^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

Here every term is positive, except when $w^2\rho<v^2$. Treating this as an equality, and writing $w^2\rho/v^2\equiv y$, we get a cubic equation in y. Let the roots of this be α, β, γ in order of magnitude. The inequality may then be written

$$(y-\alpha)(y-\beta)(y-\gamma)>0$$

Hence, either $y>\alpha$ or between β and γ.

When the central hollow is very small, x is also very small. If $x=0$

$$b=-\frac{2\rho}{1+\rho}v$$

$$c=-\frac{1}{1+\rho}\{n(n+1)w^2\rho+(n^2-3n+4)v^2\}$$

$$d=\ \frac{2n}{1+\rho}\{(n+1)w^2\rho+(n-3)v^2\}$$

In this put for the moment $n(n+1)w^2\rho/v^2+n^2-3n=2\xi$

Then

$$c=-\frac{2}{1+\rho}(\xi+2)v^2$$

$$d=\frac{4}{1+\rho}\xi v^2$$

and the above condition becomes

$$\frac{2}{1+\rho}(\xi+2)^3+\frac{\rho^2(\xi+2)^2}{(1+\rho)^2}+\frac{18\rho}{1+\rho}\xi(\xi+2)-27\xi^2+\frac{8\rho^3}{(1+\rho)^2}\xi>0$$

or

$$\xi^3-\left(8+4\rho-\frac{1}{2(\rho+1)}\right)\xi^2+\left(14+16\rho+4\rho^2-\frac{2}{1+\rho}\right)\xi+\frac{4(4+4\rho+\rho^2)}{2(1+\rho)}>0$$

whence

$$(\xi-4-2\rho)^2\left(\xi+\frac{1}{2(\rho+1)}\right)>0.$$

In this case the two roots α and β of the cubic in y become equal, and the condition of stability is

$$n(n+1)w^2\rho+n(n-3)v^2+\frac{v^2}{\rho+1}>0$$

This is always the case when $n\geqq 3$.

When $n=2$

$$6w^2\rho \overset{=}{>} \left(2-\frac{1}{\rho+1}\right)v^2$$

or

$$w^2\rho > \frac{2\rho+1}{6(\rho+1)}v^2 \qquad \qquad (68)$$

When $n=1$

$$2w^2\rho > \left(2-\frac{1}{\rho+1}\right)v^2$$

or

$$w^2\rho > \frac{2\rho+1}{2(\rho+1)}v^2 \qquad \qquad (69)$$

To $n=1$ corresponds a vibration of the ring as a whole about its state of steady motion, whereby it describes a circle, *without change in cross section,* round its mean position. Instability therefore for $n=1$ actually means instability in steadiness, whereas instability for $n=2$ means instability of form.

For a ring of the same density as the fluid, and just at the point of forming its hollow, the condition of stability of steadiness is $w^2 > \frac{3}{4}v^2$, and for stability of form $w^2 > \frac{1}{4}v^2$.

For a ring with a very dense core, the corresponding conditions are $w^2\rho > \frac{1}{2}v^2$ and $w^2\rho > \frac{1}{6}v^2$.

When the hollow is very large x is nearly equal to unity. In the case $x=1$

$$b = -nv$$

$$c = -n(n+1)w^2$$

$$d = \frac{n^2}{\rho}\{(n+1)w^2\rho - 2v^2\}v$$

and the equation to find $\lambda/v=y$ becomes, writing $w^2/v^2=z$

$$y^3 - ny^2 - n(n+1)zy + n^2\left\{(n+1)z - \frac{2}{\rho}\right\} = 0 \qquad \qquad (70)$$

When ρ is infinitely large (density of core infinitely small) the roots of this are n, and $\pm\sqrt{\{n(n+1)z\}}$. The last two agree with the result obtained for a hollow only, as might be expected. Hence, when the density of the core is small the ring is stable when its aperture is very large.

When $\rho=1$ and $z=1$ (the ordinary case treated), the roots are $-n$ and $n\pm\sqrt{n}$. This is, therefore, also stable when the aperture is large.

The condition of the reality of the roots of equation (70) is

$$4n^3(n+1)^3z^3+n^4(n+1)^2z^2+18n^4(n+1)\left\{(n+1)z-\frac{2}{\rho}\right\}z+4n^5\left\{(n+1)z-\frac{2}{\rho}\right\}$$
$$>27n^4\left\{(n+1)z-\frac{2}{\rho}\right\}^2$$

or

$$(n+1)^3z^3-2n(n+1)^2z^2+n(n+1)\left(n+\frac{18}{\rho}\right)z-\frac{2n^2}{\rho}-\frac{27n}{\rho^2}>0$$

For all values of n. This may be written

$$n^3z(z-1)^2+n^2\left\{3z^3-4z^2+\left(1+\frac{18}{\rho}\right)z-\frac{2}{\rho}\right\}+n\left\{3z^3-2z^2+\frac{18}{\rho}z-\frac{27}{\rho^2}\right\}+z^3>0 \quad (71)$$

Call this expression Y_n.
Then the condition for stability of steadiness is $Y_1>0$, or

$$z(z-1)^2+\left\{z(3z-1)(z-1)+\frac{18}{\rho}z-\frac{2}{\rho}\right\}+\left\{3z^3-2z^2+\frac{18}{\rho}z-\frac{27}{\rho^2}\right\}+z^3>0$$

and

$$Y_n-Y_1=(n-1)\left[(n^2+n+1)z(z-1)^2+(n+1)\left\{z(3z-1)(z-1)+\frac{18}{\rho}z-\frac{2}{\rho}\right\}\right.$$
$$\left.+3z^3-2z^2+\frac{18}{\rho}z-\frac{27}{\rho^2}\right]$$
$$=(n-1)\left[(n^2+n)z(z-1)^2+n\left\{z(3z-1)(z-1)+\frac{18}{\rho}z-\frac{2}{\rho}\right\}-z^3+Y_1\right]$$

Here $n=$ or >2. Hence, if $\rho\overset{=}{>}1$

$$Y_n-Y_1\overset{=}{>}(n-1)[11z^2+16z+4+Y_1]$$

and Y_1 is positive. Hence $Y_n>Y_1$. It is therefore only needful to determine the condition of stability when $n=1$, which is given above. It is

$$8z^3-8z^2+2\left(1+\frac{18}{\rho}\right)z-\frac{2}{\rho}-\frac{27}{\rho^2}>0\,.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad(72)$$

When $\rho=10^{-1}$ this is

$$(z-5.2\ldots)(z^2+4.2\ldots z+6.5\ldots)>0$$

for $\rho=10^{-2}$

$$(z-28)(z^2+27z+1206)>0$$

and in general when ρ is very small $=10^{-p}$

$$z=\tfrac{3}{2}\times 10^{\frac{2p}{3}}$$

or

$$w/v=10^{\frac{p}{3}}\sqrt{\tfrac{3}{2}}=1{\cdot}22\times 10^{\frac{p}{3}}$$

Coming back to the general equation of condition, it is clear that, since $w^6\rho^3$ occurs on the left with a positive coefficient, whereas the highest on the right is $w^4\rho^2$, with a given value of x, it is possible to satisfy the condition of stability in the state of the ring defined by x, by taking $w^2\rho$ sufficiently large. If then $w^2\rho$ be very large, the ring would be stable up to a considerable value of x, but as the aperture increased, a point might be reached at last where it would become unstable. These considerations apply to any value of ρ. When x is very small the equation for λ/v becomes (writing $y=\lambda/v$, $w^2\rho/v^2=z^2$)

$$y^3-\frac{2\rho}{1+\rho}y^2-\frac{1}{1+\rho}\{n(n+1(z^2+(n^2-3n+4)\}y+\frac{2n}{1+\rho}\{(n+1)z^2+n-3\}=0$$

One root of this is 2 ; the others are given by

$$y^2+\frac{2}{1+\rho}y-\frac{n}{1+\rho}\{(n+1)z^2+n-3\}=0$$

which gives the roots already found in Case II., as was to be expected. The solid ring has two periods, whereas just as a hollow begins to be formed it suddenly possesses two additional ones, given by $\lambda=0$, and $\lambda=2$. It will be interesting to see how these are introduced. To do this we must have recourse to equations (62a) and determine the amplitudes of the inner and outer vibrations respectively corresponding to any given value of λ.

Taking the second of the equations (62a) and putting therein $u=0$, $x=$ small, we get

$$q\mathrm{L}\lambda(\lambda-nv)-\mathrm{M}\{(1+\rho)\lambda^2+2v\lambda-n(n+1)w^2\rho-n(n-3)v^2\}=0$$

with the value $q=0$. Hence $\mathrm{M}=0$ unless λ is one of the roots (65). In other words, as the hollow forms the outside vibrations are not immediately affected, but the internal vibrations introduce two periods peculiar to themselves, one standing vibrations ($\lambda=0$) and another ($\lambda=2v$) travelling in the direction of the cyclic motion with velocity $2v/n=2\mathrm{U}/rn$, where r is the radius of the outside surface.

19. In the general case

$$\{p\lambda^2+\mathrm{A}a\lambda-n(np+1)u^2+n\mathrm{A}au\}\{(p+\rho)\lambda^2-\mathrm{A}a\lambda-n(np-1)v^2-n\mathrm{A}av-n(n+1)w^2\rho\}$$
$$-q^2(\lambda^2-n^2u^2)(\lambda^2-n^2v^2)=0$$

When reduced this is the biquadratic equation

$$\lambda^4+b\lambda^3+c\lambda^2+d\lambda+e=0$$

where

$$b=\mathrm{A}a\rho/(1+p\rho)$$

$$\begin{aligned}c=-\{(n^2+np+n(np+1)\rho)u^2+(n^2-np)v^2+np(n+1)w^2\rho\\ -n(p+\rho)\mathrm{A}au+np\mathrm{A}av+\mathrm{A}^2a^2\}/(1+p\rho)\end{aligned}$$

$$d=\mathrm{A}a\{n(np+1)u^2-n(np-1)v^2-n\mathrm{A}a(u+v)-n(n+1)w^2\rho\}/(1+p\rho)$$

$$\begin{aligned}e=n^2[(n^2-1)u^2v^2+(n+1)(np+1)u^2w^2\rho+\mathrm{A}au\{(np+1)uv-(np-1)v^2\\ -(n+1)w^2\rho\}-\mathrm{A}^2a^2uv]/(1+p\rho)\end{aligned}$$

Now

$$\mu'=-4\pi\mathrm{A}a^3(k_2{}^2-k_1{}^2)=-4\pi\mathrm{A}a^3k_2{}^2(1-x)$$

$$v=\frac{\mu_1+\mu'}{8\pi a^2k_2{}^2}=\frac{\mu_1}{8\pi a^2k_2{}^2}-\tfrac{1}{2}\mathrm{A}a(1-x)$$

$$u=\frac{\mu_1}{8\pi a^2k_1{}^2}=\frac{\mu_1}{8\pi a^2k_2{}^2x}$$

therefore

$$v=xu-\tfrac{1}{2}\mathrm{A}a(1-x)$$

or

$$\mathrm{A}a=\frac{2}{1-x}(xu-v)$$

therefore

$$\frac{\mathrm{A}a}{v}=-\frac{2}{1-x}\cdot\frac{\mu'}{\mu_1+\mu'}$$

$$\frac{u}{v}=\frac{\mu_1}{\mu_1+\mu'}\frac{k_2{}^2}{k_1{}^2}=\frac{\mu_1}{x(\mu_1+\mu'}$$

$$\frac{w}{v}=\frac{\mu_2}{\mu_1+\mu'}$$

If then $\lambda/v=y$, the equation for y is

$$y^4+by^3+cy^2+dy+e=0$$

where now (writing $\mu'/(\mu_1+\mu')$, &c. $=\mu'$, &c.)

$$\left.\begin{aligned}
b&=-\frac{2\mu'\rho}{(1-x)(1+p\rho)}\\
c&=-\left\{(n^2+np+n(np+1)\rho)\frac{\mu_1^2}{x^2}+n^2-np+np(n+1)\rho\mu_2^2\right.\\
&\qquad\left.+2n\frac{p+\rho}{x(1-x)}\mu_1\mu'-\frac{2np}{1-x}\mu'+\frac{4\mu'^2}{(1-x)^2}\right]\frac{1}{1+p\rho}\\
d&=\frac{2\mu'n}{(1-x)(1+p\rho)}\left\{(np-1)-(np+1)\frac{\mu_1^2}{x^2}+(n+1)\mu_2^2\rho-\frac{2\mu'}{1-x}\left(1+\frac{\mu_1}{x}\right)\right\}\\
e&=\frac{n^2}{1+p\rho}\left[(n^2-1)\frac{\mu_1^2}{x^2}+(n+1)(np+1)\frac{\mu_1^2\mu_2^2}{x^2}\rho-\frac{2\mu'\mu_1}{x(1-x)}\left\{(np+1)\frac{\mu_1}{x}-np+1\right.\right.\\
&\qquad\left.\left.-(n+1)\mu_2^2\rho\right\}-\frac{4\mu_1\mu'^2}{x(1-x)^2}\right]
\end{aligned}\right\}\tag{73}$$

The general investigation of the question of stability would be very laborious, but it may be useful to put down the values of the coefficients when the aperture becomes very large, or $x=1$. They are

$$\left.\begin{aligned}
b&=-n\mu'\\
c&=-n(n+1)\mu_2^2-\left(n^2+\frac{n}{\rho}\right)\mu_1\\
d&=\frac{n^2\mu'}{\rho}\{(n+1)\mu_2^2\rho-\mu_1(1+\mu_1)-2\}\\
e&=n^3\mu_1\left\{(n+1)\mu_1\mu_2^2-\frac{2\mu'}{\rho}(1+\mu_1)\right\}
\end{aligned}\right\}\quad\ldots\ldots\ldots\tag{74}$$

In which

$$\mu_1+\mu'=1$$

20. *Pulsations.*—Let the pulsations be given by

$$k=k_1(1+\xi)\qquad k=k_2(1+\eta)$$

Then since the volume of the core remains unaltered

$$k_1^2\xi=k_2^2\eta.$$

Since the stream function is now many-valued, it will be advisable to use the potential function for the part of the motion due to this: denoting it by ϕ, it is of the form

$$\phi=\sqrt{C-c}\{AP_0+BQ_0\}$$

where A and B are constants determined by the equation

$$\frac{\delta\phi}{\delta u}\frac{du}{dn} = \text{vel. inwards at the two surfaces}$$

$$=\frac{dn}{dt}=\frac{dn}{du}\cdot\frac{du}{dk}\dot{k}=-\frac{dn}{du}(\dot{\xi} \text{ and } \dot{\eta})$$

Hence

$$\frac{\delta\phi}{\delta u}=-\frac{a^2}{(\mathrm{C}-c)^2}\dot{\xi}=-4a^2k^2\dot{\xi}$$

Now

$$\frac{\delta\phi}{\delta u}=\frac{\mathrm{S}}{2\sqrt{(\mathrm{C}-c)}}(\mathrm{AP}_0+\mathrm{BQ}_0)+\sqrt{(\mathrm{C}-c)}\left\{\mathrm{A}\frac{d\mathrm{P}_0}{du}+\mathrm{B}\frac{d\mathrm{Q}_0}{du}\right\}$$

$$=\frac{\mathrm{A}}{2\sqrt{(\mathrm{C}-c)}}\left\{\mathrm{SP}_0+2(\mathrm{C}-c)\frac{d\mathrm{P}_0}{du}\right\}+\frac{\mathrm{B}}{2\sqrt{(\mathrm{C}-c)}}\left\{\mathrm{SQ}_0+2(\mathrm{C}-c)\frac{d\mathrm{Q}_0}{du}\right\}$$

$$=\mathrm{A}\sqrt{2} \qquad \text{to the lowest order.}$$

Hence

$$\mathrm{A}\sqrt{2}=-4a^2k_1^{\,2}\dot{\xi}=-4a^2k_2^{\,2}\dot{\eta}$$

Therefore

$$\phi=-2\sqrt{2}a^2k_1^{\,2}\sqrt{(\mathrm{C}-c)}.\dot{\xi}\mathrm{P}_0\, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (75)$$

Again the circulation is unaltered. A term must therefore be added to ψ; let it be denoted by χ. Then χ is of the form

$$\chi=\frac{1}{\sqrt{(\mathrm{C}-c)}}\Sigma\{\mathrm{A}_0\mathrm{R}_0+\mathrm{B}_0\mathrm{T}_0+\ldots\}$$

The circulation of this is

$$-\frac{\pi\sqrt{2}}{a}\Sigma\mathrm{A}_0$$

But the circulation of ψ round the new boundary is

$$\left(\mathrm{U}+\frac{d\mathrm{U}}{dk}dk\right)2\pi r.$$

Along the inner boundary this is

$$=4\pi ak_1\left\{\mathrm{U}_1+\mathrm{U}_1\xi+\frac{d\mathrm{U}}{dk}k_1\xi\right\}$$

$$=4\pi ak_1\mathrm{U}_1-8\pi\mathrm{A}a^3k_1^{\,2}\xi \quad \text{by (61)}$$

Hence since the circulation is unaltered

$$-8\pi\mathrm{A}a^3k_1^{\,2}\xi-\frac{\pi\sqrt{2}}{a}\mathrm{A}_0=0$$

$$\mathrm{A}_0=-4\sqrt{2}\mathrm{A}a^4k_1^{\,2}\xi=-4\sqrt{2}\mathrm{A}a^4k_2^{\,2}\eta \quad . \quad . \quad . \quad . \quad . \quad . \qquad (76)$$

For the pressure condition we require the stream function. Now $2\pi\psi$ is the flux across any diaphragm with circular boundary through the point in question. Take it to be the annulus $v=\pi$, from $u=u_1$ to $u=u$, and then the portion of the tore $u=u$ from $v=\pi$ to $v=v$. We need only to find the part of ψ due to ϕ. Now ϕ produces no motion across the annulus. Hence

$$2\pi\psi'=\int_v^\pi -\frac{\delta\phi}{\delta u}\frac{du'}{dn}\cdot 2\pi\rho\frac{dn'}{dv}\,dv$$

or

$$\begin{aligned}\psi'&=-a\int_v^\pi\frac{\delta\phi}{\delta u}dv\\&=4a^3k_1^2\dot{\xi}(\pi-v)\\&=4a^3k_2^2\dot{\eta}(\pi-v)\qquad(77)\end{aligned}$$

which is many-valued, as it ought to be.

Again, we need the flow due to χ. This is

$$\begin{aligned}\phi'&=\int_{u_1}^u\left[\frac{1}{\rho}\frac{\delta\chi}{dv}\cdot\frac{dv}{dn'}\cdot\frac{dn}{u}\right]_{v=\pi}du+\int_v^\pi\left[\frac{1}{\rho}\frac{\delta\chi}{\delta u}\frac{du}{dn}\cdot\frac{du'}{dv}\right]_{u=u}dv\\&=\int_{u_1}^u\left[\frac{1}{\rho}\frac{\delta\chi}{\delta v}\right]_{v=\pi}du+\int_v^\pi\left[\frac{1}{\rho}\frac{\delta\chi}{\delta u}\right]_{u=u}dv\\&=\int_v^\pi\left[\frac{1}{\rho}\frac{\delta\chi}{\delta u}\right]_{u=u}dv\text{ to lowest order.}\end{aligned}$$

$$\begin{aligned}\phi'&=\frac{A_0}{a}\int_v^\pi\left\{-\frac{S}{2(C-c)^{\frac{3}{2}}}R_0-\frac{1}{4(C-c)^{\frac{1}{2}}}SP_0\right\}dv\\&=-\frac{A_0}{2a}\left\{-\surd(2)(\tfrac{1}{2}L-1)+\frac{L}{\surd 2}\right\}(\pi-v)\\&=-\frac{A_0}{a\surd 2}(\pi-v)\\&=4Aa^3k_1^2\xi(\pi-v)=4Aa^3k_2^2\eta(\pi-v)\qquad(78)\end{aligned}$$

The pressure is given by

$$\begin{aligned}\frac{p}{d}=F(t)-\dot{\phi}-\dot{\phi}'-\tfrac{1}{2}\left(U+\frac{dU}{dk}\delta k+\frac{1}{\rho}\frac{\delta\chi}{du}\frac{du}{dn}-\frac{\delta\phi}{\delta v}\frac{dv}{dn}\right)^2\\-\tfrac{1}{2}\left(\frac{1}{\rho}\frac{\delta\psi}{\delta v}\frac{dv}{dn}+\frac{\delta\phi}{du}\frac{du}{dn}\right)^2+A\left(\psi+\frac{\delta\psi}{\delta k}\delta k+\psi'+\chi\right)\\=F(t)-\tfrac{1}{2}U^2+A\psi-\dot{\phi}-\dot{\phi}'-U\frac{dU}{dk}\delta k-\frac{U}{2a^2k}\frac{\delta\chi}{\delta u}+A\left(\frac{\delta\psi}{\delta k}\delta k+\psi'+\chi\right)\end{aligned}$$

keeping only the most important terms. Substituting the values already obtained

$$\begin{aligned}\frac{p}{d}=&\mathrm{F}(t)-\tfrac{1}{2}\mathrm{U}^2+\mathrm{A}\psi\\&+2a^2k_1^2\frac{\mathrm{P}_0}{\sqrt{k}}\ddot{\xi}-4\mathrm{A}a^3k_1^2\xi(\dot{\pi}-v)\\&+\mathrm{U}(\mathrm{U}+2\mathrm{A}a^2k)\xi-2\mathrm{U}\mathrm{A}a^2k_1\xi\\&+\mathrm{A}\{-2a^2k\mathrm{U}\xi+4a^3k_1^2\xi(\dot{\pi}-v)+4\mathrm{A}a^4k_1^2(\mathrm{L}-2)\xi\}\\=&\mathrm{F}(t)-\tfrac{1}{2}\mathrm{U}^2+\mathrm{A}\psi+4a^2k_1^2\mathrm{L}\ddot{\xi}+\mathrm{U}(\mathrm{U}-2\mathrm{A}a^2k)\xi+4\mathrm{A}^2a^4k_1^2(\mathrm{L}-2)\xi\end{aligned}$$

Hence along the inside

$$\mathrm{F}(t)-\tfrac{1}{2}\mathrm{U}_1^2+\mathrm{A}\psi_1+4a^2k_1^2\mathrm{L}_1\ddot{\xi}+4a^2\left\{\mathrm{A}^2(\mathrm{L}_1-2)a^2+\left(\frac{\mathrm{U}_1}{2ak_1}\right)^2-\frac{\mathrm{U}_1}{2ak_1}\mathrm{A}a\right\}k_1^2\xi=0$$

Whilst along the outside boundary

$$\begin{aligned}&d\left\{\mathrm{F}t-\tfrac{1}{2}\mathrm{U}^2_2+\mathrm{A}\psi_2+4a^2k_1^2\mathrm{L}_2\ddot{\xi}+4a^2\left\{\mathrm{A}^2(\mathrm{L}_2-2)a^2+\left(\frac{\mathrm{U}_2}{2ak_2}\right)^2-\frac{\mathrm{U}_2}{2ak_2}\mathrm{A}a\right\}k_1^2\xi\right.\\&\qquad=d'\left\{4a^2\mathrm{L}_2k_1^2\ddot{\xi}+4a^2\left(\frac{\mathrm{U}'_2}{2ak_2}\right)^2k_1^2\xi\right\}\end{aligned}$$

Remembering that the large terms have been already annulled, we get by subtracting the two equations and writing ξ for $k_1^2\xi$, &c., as before

$$\begin{aligned}(\mathrm{L}_2-\mathrm{L}_1)\ddot{\xi}+\{\mathrm{A}^2a^2(\mathrm{L}_2-\mathrm{L}_1)+v^2-u^2-\mathrm{A}a(v-u)\}\xi\\=\frac{d'}{d}\{\mathrm{L}_2\ddot{\xi}+w^2\xi\}\end{aligned}$$

or

$$\left\{\mathrm{L}_1+\left(\frac{d'}{d}-1\right)\mathrm{L}_2\right\}\ddot{\xi}+\left\{w^2\frac{d'}{d}-(v-u)(v+u-\mathrm{A}a)+\mathrm{A}^2a^2(\mathrm{L}_1-\mathrm{L}_2)\right\}\xi=0 \quad . \quad (79)$$

whence (if $\rho=d'/d$). The time of pulsation is

$$2\pi\sqrt{\left\{\frac{\mathrm{L}_2(\rho-1)+\mathrm{L}_1}{\rho w^2+\mathrm{A}^2a^2(\mathrm{L}_1-\mathrm{L}_2)-(v-u)(v+u-\mathrm{A}a)}\right\}} \quad . \quad . \quad . \quad . \quad . \quad (80)$$

The numerator of this is always positive. Hence the condition for stability is that

$$\rho w^2+\mathrm{A}^2a^2(\mathrm{L}_1-\mathrm{L}_2)+(v-u)\mathrm{A}a>v^2-u^2$$

When the added circulation is in the same direction as that of the core—as has been supposed throughout, $v>u$.

If there are no added circulations, $u=0$, $w=v$, and

$$v=-\tfrac{1}{2}\mathrm{A}a\left(1-\frac{k_1^2}{k_2^2}\right)=-\tfrac{1}{2}\mathrm{A}a(1-x)$$

and the condition of stability is

$$\rho>1+\frac{2}{1-x}-\frac{2\log 1/x}{(1-x)^2}$$

The maximum value of the right hand member is when

$$\frac{1+x}{x(1-x)^2}+\frac{2\log x}{(1-x)^3}=0$$

or

$$1-x^2+2x\log x=0$$

whence $x=1$, which makes the condition become

$$\rho>0.$$

Now x is always <1, and since ρ is always positive, the condition is satisfied.

In general

$$w=\frac{\mathrm{U}_2'}{2\pi ak_2}=\frac{\mu_2}{8\pi a^2k_2^2}$$

$$v=\frac{\mu_1+\mu'}{8\pi a^2k_2^2}$$

$$u=\frac{\mu_1}{8\pi a^2k_1^2}$$

Also

$$\mu'=-4\pi\mathrm{A}a^3(k_2^2-k_1^2)$$

Hence condition of stability is that

$$\begin{aligned}\mu_2^2\rho&>\left(\mu_1+\mu'-\mu_1\frac{k_2^2}{k_1^2}\right)\left(\mu_1+\mu'+\mu_1\frac{k_2^2}{k_1^2}+2\mu'\frac{k_2^2}{k_2^2-k_1^2}\right)-4\mu'^2\left(\frac{k_2}{k_2-k_1}\right)^2\log\frac{k_2}{k_1}\\&>\left\{\mu_1\left(1-\frac{1}{x}\right)+\mu'\right\}\left\{\mu_1\left(1+\frac{1}{x}\right)+\mu'\right\}+2\mu'\left\{\left(\mu_1\left(1-\frac{1}{x}\right)+\mu'\right)\frac{1}{1-x}-\mu'\frac{\log 1/x}{(1-x)^2}\right\}.\end{aligned}$$

When x is small the right hand side is evidently negative and the condition is satisfied. When $x=1$ the right hand member is 0.

Consequently

$$\mu_2^2\rho>0$$

In general

$$\mu_2^2\rho > \mu_1^2\left(1-\frac{1}{x^2}\right)+2\mu_1\mu'\left(1-\frac{1}{x}\right)+\mu'^2\left(1+\frac{2}{1-x}+\frac{2\log x}{(1-x)^2}\right)$$

all these terms are negative. Hence the core is always stable for pulsations.

Substituting for u, v, w the time of pulsation is given by

$$16\pi^2a^2k_2^2\sqrt{\left\{\frac{\rho L_2+\frac{1}{2}\log 1/x}{\mu_2^2\rho-\mu_1^2\left(1-\frac{1}{x^2}\right)-2\mu_1\mu'\left(1-\frac{1}{x}\right)+\mu'^2\left(1+\frac{2}{1-x}+\frac{2\log x}{(1-x)^2}\right)}\right\}}\ . \quad (81)$$

For a coreless ring this is

$$\frac{16\pi^2a^2k^2}{\mu_2}\sqrt{\log\frac{4}{k}}$$

which agrees with the value obtained in [I. § 14].

[Added April 3, 1886.—The cubic giving the times of vibration of a hollow ring of the same density as the fluid, and with no extra circulations, can be solved. The equation is

$$y^3-\frac{1-x^n}{1-x}y^2-\left\{n^2-\frac{n(1+x^n)}{1-x}+\frac{2(1-x^n)}{(1-x)^2}\right\}y+\frac{2n}{1-x}\left(n-\frac{1-x^n}{1-x}\right)=0.$$

The roots are

$$y=-n,\quad \tfrac{1}{2}\left(n+\frac{1-x^n}{1-x}\right)\pm\tfrac{1}{2}\sqrt{\left\{\left(n+\frac{1-x^n}{1-x}\right)^2-\frac{8}{1-x}\left(n-\frac{1-x^n}{1-x}\right)\right\}}$$

The corresponding times of vibration are $\frac{4\pi^2r^2}{\mu y}$, where r is the radius of the section.]

ERRATA IN PAPER ON "STEADY MOTION AND SMALL VIBRATIONS OF A HOLLOW VORTEX," PHIL. TRANS., VOL. 175.

1. Page 188, line 6 from bottom, the coefficient of k^2 in U is wrong, since the full value of ψ_0 was not substituted.
2. Page 191, line 4, *for* $1-2k^2$ *read* $1+2k^2$.
3. „ „ 6, *for* $(L-\frac{5}{2})$ *read* $L-\frac{1}{2}$.
4. „ „ 7, the coefficient of k^2 is $\frac{3}{2}(L-\frac{1}{2})^2$.
5. In § 13 the effect of the surface velocity in modifiying the normal motion of the wave motion has been neglected. The time of vibration there given is therefore wrong. The correct value is given in the foregoing discussion in § 17.